Ellipse fitting using orthogonal hyperbolae and Stirling's oval

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Abstract

Two methods for approximating the normal distance to an ellipse using a) its orthogonal hyperbolae, and b) Stirling's oval are described. Analysis with a set of quantitative set of measures shows that the former provides an accurate approximation with little irregularities or biases. Its suitability is evaluated by comparing several approximations as error of fit functions and applying them to ellipse fitting.

1 Introduction

A common goal in image analysis is to find the best fitting ellipse to a set of data points. This enables a higher level representation of, for example, edge data, which is useful for many applications of computer vision. A large body of work has been developed on ellipse fitting techniques, mostly using least squared error [1, 2] but also other criteria such as the least median of squares [7, 8]. The majority of these fitting methods operate by minimising some function of the errors between the data points and the ellipse. Although the Euclidean distance along the normal between the point and the ellipse would be well suited for an error function it requires solving a quartic equation. Therefore, more efficiently computable approximations to this distance are usually used instead, some examples of which are given in [3, 4, 9, 10, 11].

Recently we have analysed the accuracies and inherent biases of several such approximations [6, 5]. The best method used the focal property of ellipses. Given an ellipse with foci $\mathbf{F}$ and $\mathbf{F}'$, and a point $\mathbf{P}$ on the ellipse, then the lines $\mathbf{FP}$ and $\mathbf{F}'\mathbf{P}$ make equal angles with the tangent to the ellipse at $\mathbf{P}$. Thus the angular bisector of $\mathbf{FP}$ and $\mathbf{F}'\mathbf{P}$ is the normal to the ellipse at $\mathbf{P}$. Although this does not hold when $\mathbf{P}$ lies off the ellipse, when the angular bisector is taken as an approximation to the normal then good results displaying little curvature bias, asymmetry, or non-linearity were obtained. More details on the definition and calculation of the error assessment measures are given in Rosin [6].

In this paper we describe two new techniques for estimating the perpendicular distance to an ellipse. They are based on complementary approaches: the first approximates the normal itself using a hyperbola and the intersection with the true ellipse is obtained, while the second approximates the ellipse using circular arcs to which the true normal can be determined.

2 Orthogonal Conics

Families of ellipses and hyperbolae which are confocal are mutually orthogonal, as shown in figure 1. Given that much of a hyperbola is "fairly" straight then the confocal hyperbola passing through $\mathbf{P}$ should be a reasonable approximation of the straight line through $\mathbf{P}$ that is normal to the ellipse. The three stages in its calculation are as follows: first, the unique hyperbola H that is confocal with the ellipse E and passes through $\mathbf{P}$ is determined. Next the four points of intersection $\mathbf{I}$ of E and H are calculated. Finally, rather than actually use the arc length along the hyperbola, the Euclidean distance of the normal from $\mathbf{P}$ to E is approximated by the Euclidean distance from $\mathbf{P}$ to $\mathbf{I}$.

To simplify the equations the ellipse and data are transformed into the canonical coordinate frame. The canonical equations for an ellipse and hyperbolae are

$$\frac{x^2}{a_e^2} + \frac{y^2}{b_e^2} = 1 \tag{1}$$

and

$$\frac{x^2}{a_h^2} - \frac{y^2}{b_h^2} = 1. \tag{2}$$

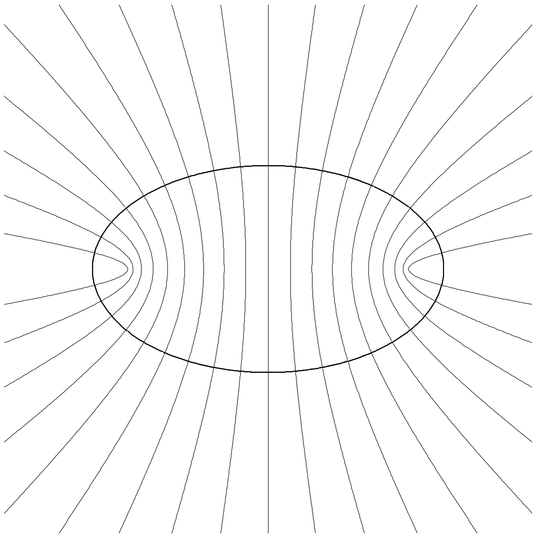


Figure 1: Orthogonal ellipse and hyperbolae

The position of the foci along the X axis of the ellipse and hyperbolae are defined as

$$f_e = \sqrt{a_e^2 - b_e^2}$$

and

$$f_h = \sqrt{a_h^2 + b_h^2} \tag{3}$$

where a_e, b_e and a_h, b_h are the major and minor axes of the ellipse and hyperbolae respectively. Since we will only be considering confocal conics then $f_e = f_h$. To determine the parameters of the confocal hyperbola that passes through **P** we use the following substitutions

$$\begin{aligned} A &= a_h^2 \\ F &= f_h^2 = f_e^2 \\ X &= x^2 \\ Y &= y^2 \end{aligned} \tag{4}$$

and rewrite (2) as

$$\frac{X}{A} - \frac{Y}{F - A} = 1$$

which produces a quadratic in A

$$A^2 - A(X + Y + F) + XF = 0. \tag{5}$$

Solving (5) for A and resubstituting in (4) and (3) gives us a_h and b_h. The intersection $\mathbf{I} = (x_i, y_i)$ of the ellipse and hyperbola is found by solving the simultaneous equations (1) and (2), to get

$$\begin{aligned} x_i &= \pm a_h \sqrt{\frac{a_e^2\left(b_e^2 + b_h^2\right)}{a_h^2 b_e^2 + a_e^2 b_h^2}} \\ y_i &= \pm \frac{b_e b_h \sqrt{a_e^2 - a_h^2}}{\sqrt{a_h^2 b_e^2 + a_e^2 b_h^2}} \end{aligned}$$

and the calculation of $\overline{\mathbf{PI}}$ is now straightforward. Four solutions are obtained, and the shortest distance selected.

3 Stirling's Oval

We now describe a second approach using a recently published method developed by James Stirling around 1744 for approximating an ellipse using four circular arcs [12]. The resulting oval provides a good

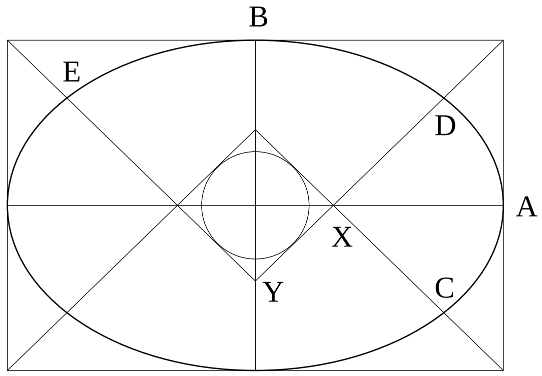


Figure 2: Stirling's circular arc approximation of an ellipse

approximation if the ellipse is not too eccentric, appearing virtually indistinguishable if $\frac{a}{b} < 2$. Moreover, the arcs join with C^1 continuity.

The oval can be generated by drawing the ellipse's minimum bounding rectangle as shown in figure 2. The lines drawn from the corners are tangent to a central circle, and their intersections with the axes define the centres of the approximating circular arcs. That is, the centres of arcs CD and DE are X and Y respectively, and the two opposite arcs are defined similarly. The centres $X = (x, 0)$ and $Y = (0, -y)$ can be calculated algebraically:

$$\begin{aligned} m &= \frac{a+b}{2} \\ n &= \frac{a-b}{2} \\ x &= \frac{2mn + n^2 + n\sqrt{2m^2 - n^2}}{m+n} \\ y &= \frac{2mn - n^2 + n\sqrt{2m^2 - n^2}}{m-n}. \end{aligned}$$

Calculating the perpendicular distance from a point P is now straightforward since we can easily calculate the perpendicular to the appropriate circular arc. If the ellipse and data are transformed to the canonical position (centred at the origin aligned with the axes) then we can immediately transform the point into the first quadrant. Arc selection is now limited to choosing between CD and DE. P is tested to see which side of line XD it lies on. Then we calculate the distance to the arc centre, and finally subtract the radius, so that the distance can be written as

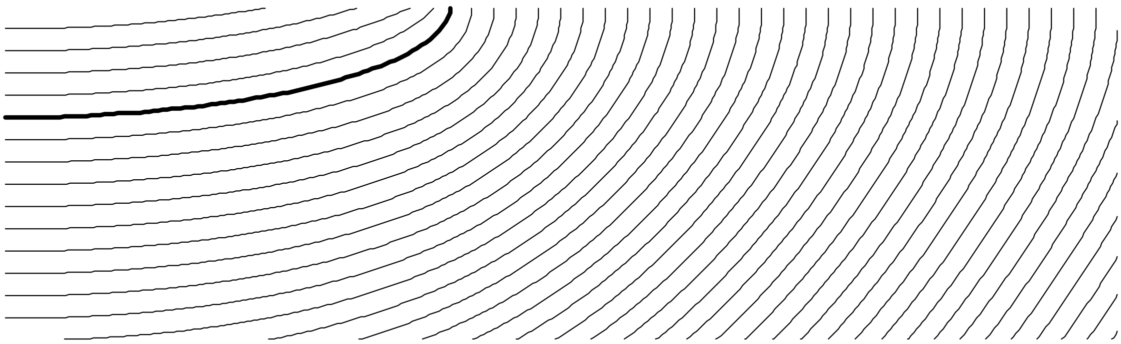
$$d = \begin{cases} |PX| - (a - x) & \text{if } y_T + \frac{y}{x}(x_T - x) < 0 \\ |PY| - (b + y) & \text{otherwise,} \end{cases}$$

where (x_T, y_T) is the transformed point.

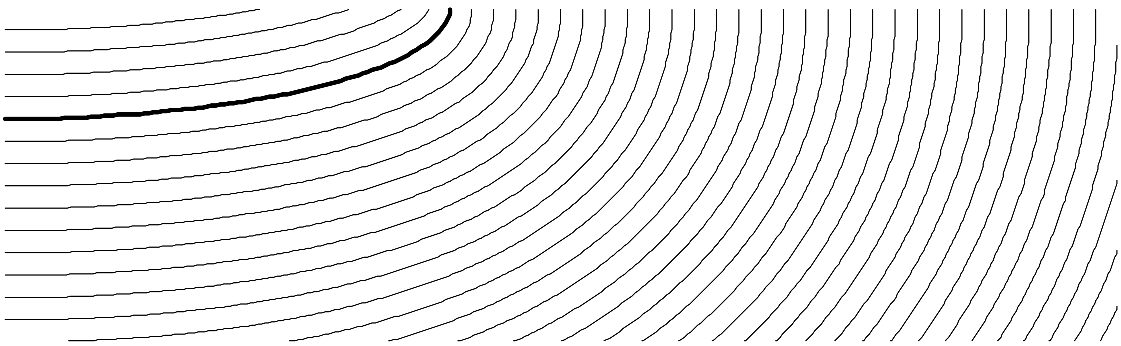
4 Experimental Results

We start by visualising the error functions by plotting their iso-value contours in figure 3. The original ellipse with axes $a = 400$ and $b = 100$ is drawn bold. Contours for the angular bisector and confocal hyperbola methods appear very similar (figure 3a and figure 3b). However, when overlaying them (contours from the angular bisector method are shown gray) the differences can be seen more clearly (figure 3c). Using Stirling's oval the discontinuity between the two circular arcs is evident (figure 3d). Also, due to the high eccentricity of the ellipse the approximation is rather poor, especially close to the boundary.

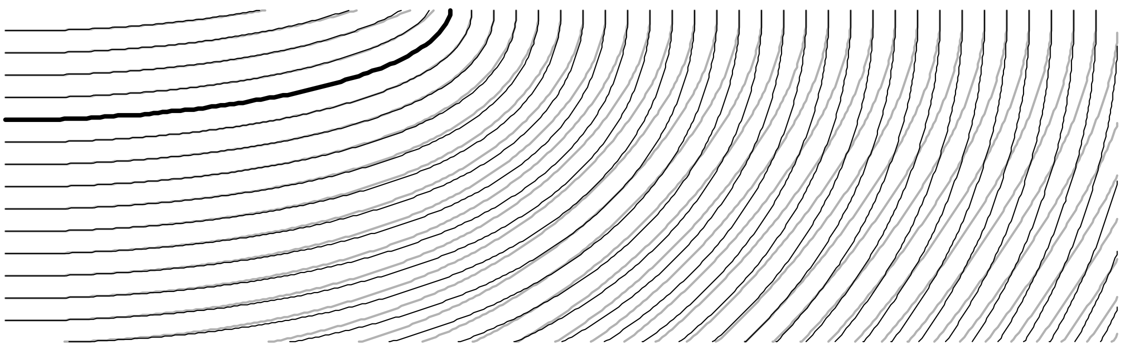
A more quantitative assessment is given in tables 1 and 2 which use different noise models to characterise the performance close to and far from the ellipse boundary respectively. EOF_1 and EOF_2 are error of fit functions often used for ellipse fitting, and are the algebraic distance and the algebraic distance divided by its gradient respectively. The angular bisector is labelled as EOF_{13}, the new orthogonal conic approach is given as EOF_{14}, and the method based on Stirling's oval is given as EOF_{15}. To aid comparison, all results are normalised against EOF_1. It can be seen that close to the boundary all the



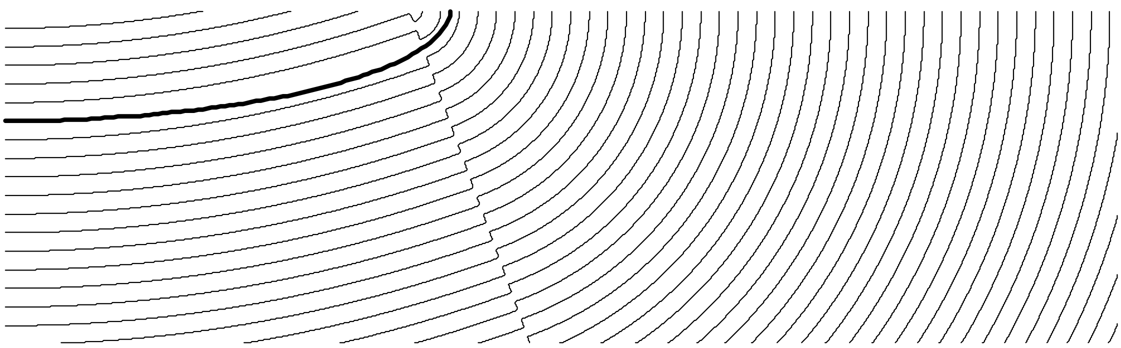
(a) angular bisector



(b) confocal hyperbola



(c) both methods overlaid



(d) Stirling's oval method

Figure 3: Iso-value contours

EOFs have similar linearity L, i.e. the error measure changes linearly with increasing distance from the ellipse. Further from the boundary EOF_2 does more poorly while $EOFs_{13-15}$ outperform EOF_1 slightly. Although EOF_2 has a lower curvature bias C than EOF_1, EOF_{13} is much better, while EOF_{14} exhibits almost no curvature bias for distant points. the performance of EOF_{15} is significantly better than EOF_1 but poorer than the rest. EOF_2 and EOF_{15} have extremely poor asymmetry A (the variation between corresponding errors values inside and outside the ellipse); the former more so far from the ellipse and the latter close to the ellipse. The overall goodness measure calculated both inside and outside the ellipse (G) or only outside the ellipse (G') also shows EOF_{13} and EOF_{14} improving upon the other methods, especially far from the boundary, while EOF_{15} receives a poor score.

Table 1: Normalised assessment results with $N(0, 2)$ noise model; $a = 400$, $b = 100$

EOF	L	C	A	G	G'
1	1.000	1.000	1.000	1.000	1.000
2	0.987	0.011	3.978	0.808	0.841
13	1.000	0.009	1.125	0.775	0.822
14	1.000	0.009	1.107	0.775	0.822
15	1.000	0.193	8.997	4.076	3.340

Table 2: Normalised assessment results with $N(0, 64)$ noise model; $e = 400$, $b = 100$

EOF	L	C	A	G	G'
1	1.000	1.000	1.000	1.000	1.000
2	0.877	0.041	8.404	2.771	0.099
13	1.006	0.002	0.747	0.007	0.009
14	1.006	0.000	0.755	0.002	0.001
15	1.005	0.118	3.276	4.347	4.749

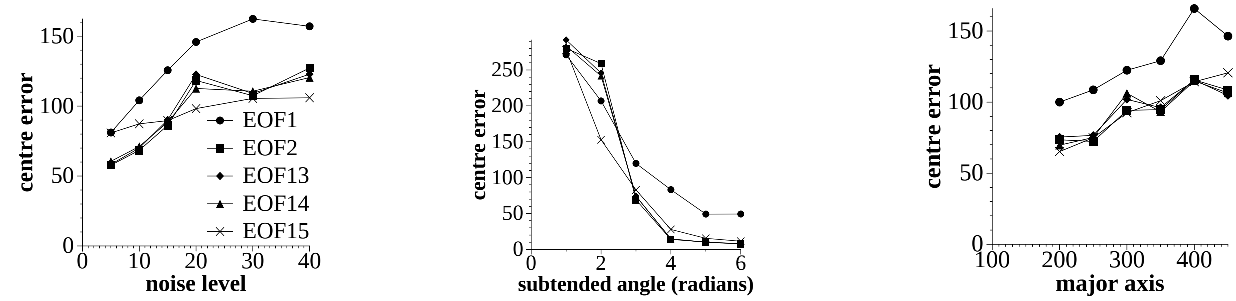
The effectiveness of the various distance approximations is tested by fitting ellipses. The Least Median of Squares (LMedS) fit is found using the different approximations as error of fit functions (see Rosin [7] for further details and examples). 4000 sets of synthetic data were generated for ellipses, each containing between 18 and 89 points, varying the following parameters: major axis $a = [200, 450]$, minor axis $b = 100$, subtended angles $\theta = [1, 6]$, and added Gaussian noise $\sigma = [5, 40]$. The alpha trimmed ($\alpha = 0.1$) mean errors are listed in table 3, and show that EOF_1 is rated worst, while EOF_{14} performs slightly better than the other methods, and EOF_{15} does not do particularly well.

Table 3: Trimmed mean errors of estimated centre coordinates by LMedS ellipse fitting

EOF	Centre Error
1	97.6
2	64.7
13	62.7
14	61.8
15	75.7

Figure 4 shows how varying the ellipse and noise parameters affect the quality of the fit. Although EOF_{15} gets an overall poor assessment it can be seen to be insensitive to noise and is competitive for small arcs. As expected, the results of fitting deteriorates for all EOFs as noise increases, subtended angle decreases, and eccentricity increases.

Table 4 shows a count of the arithmetic operations involved in calculating the distance approximations. The true Euclidean distance (obtained by solving the quartic equation) is also included as EOF_{16}. In addition, the figures for $EOFs_{15}$ include the one-off calculations to determine the oval which account for



(a) increasing Gaussian noise (b) increasing subtended angle (c) increasing eccentricity

Figure 4: Centre estimation error

half the complexity. Since the algorithms have not been carefully coded these figures should only be taken as rough estimates of the algorithms' complexities. As expected, the algebraic and weighted distances are much more efficient than the better approximations. The complexities of EOFs$_{13-15}$ are comparable, and are still substantially less than solving the quartic equation.

Table 4: Number of occurrences of arithmetic operations for distance approximations

EOF	$+-$	$\times\div$	$\surd$	trig.
1	5	8	0	0
2	6	13	1	0
13	26	41	2	8
14	25	35	6	4
15	23	20	3	4
16	52	110	7	6

5 Conclusions

We have described two new approaches for approximating the normal distance to an ellipse. Along with some other distance approximations they were analysed by a set of criteria that enable a quantitative comparison. The method using orthogonal hyperbolae proved to be the most accurate, showing good linearity and asymmetry, comparable with the confocal method we described previously, while its curvature bias and overall goodness is much improved for distant points. The approach using Stirling's oval fared poorly for eccentric ellipses as the circular approximation breaks down. When the distance approximations are applied to the task of ellipse fitting the orthogonal hyperbolae method produces slightly better fits, as measured by the trimmed mean of the errors in centre location.

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