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A FAMILY OF CONSTRUCTIONS OF APPROXIMATE ELLIPSES

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Several constructions for piecewise circular approximations to ellipses are examined. It is shown that a simple approach based on positioning the arc centres based on factors of the difference in major and minor axes lengths generates a family of tangent continuous solutions that can provide near optimal approximations for specific aspect ratios.

Keywords: ellipse, approximation, quadrarc, biarc, piecewise circular arc

1. Introduction

Ellipses have long played a part in architecture. For instance, more than 4000 years ago in the Middle Minoan period a large oval house was built in Crete [1], while up to the present day in many parts of Africa, India, and Europe oval huts and houses were commonplace [2]. The Romans embraced the ellipse, and employed it on a vast scale in their ampitheatres [3]. In the Baroque period of the Renaissance the ellipse was once again applied, mostly in the design of churches [4]. An elliptical theme recurs in Georgian architecture [5], appearing at all scales from decorative ceiling plaster details, to the plan for drawing rooms, to entire terraces of houses. Right through to modern times the ellipse has remained a popular form in architecture, and some more recent examples from the last five years are the Church of San Giovanni Battista, by Mario Botta [6], the Ruskin Library by Richard MacCormac [7], and Walsall bus station by Allford, Hall, Monaghan, and Morris [8]. Thus the ellipse remains one of the standard shape primitives (along with the circle, square, triangle, etc.) that figure prominently in ancient and modern design (architectural, engineering, graphical, etc.), and is therefore worthy of particular attention.

Although the ellipse provides more dynamic possibilities than the relatively static forms of straight lines and circles it also introduces complications. In particular, it is more difficult to lay out, and its continuously varying curvature means that for precise building construction a large range of brick shapes should be used [9]. Another consideration is that its perimeter is difficult to calculate; this is important not only for the estimation of materials, but is also necessary if a wall's length is to be subdivided at regular intervals for the placement of columns, windows, etc. A quantitative indication of this overhead is given by the Georgian author on building, Batty Langley who suggests that workmanship and materials be charged an additional 50% when constructing elliptical walls as compared to straight walls [10].

2. Approximate Constructions of the Ellipse

This brings us to the approximation of ellipses by circular arcs. We know that these approximations were extensively used by artisans, artists, architects, etc. from the Renaissance times onwards [11], and there has been considerable debate as to their earlier application by the Romans [12]. The advantage of a piecewise circular approximation is that it offers the possibility of simple construction and simple analysis (e.g. of perimeter) while retaining effectively the same shape as the ellipse. There are many areas in computer graphics where this is beneficial, e.g. rendering [13], intersections with other graphics primitives, font generation [14], drawing dashed ellipses (with constant sized dash spacing), etc.

Moreover, with circular arcs it is straightforward to generate parallels, avoiding the eighth order equations necessary for ellipses [15]. An application in architecture can be seen in the rows of seats in ampitheatres, while in CAD the offset curves of paths made by NC milling machines are another example [16].[a] Likewise, calculating normals (useful for ellipse fitting for instance [17]) is trivial for circles; in comparison a quartic needs to be solved for the ellipse.

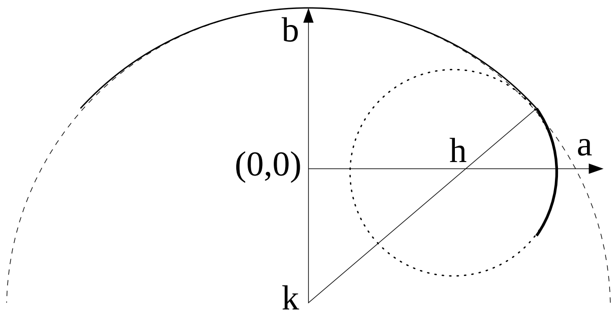


Fig. 1. Geometry of the four-arc approximate ellipse with tangent continuity at arc joins.

There are various criteria that could be considered in deciding on a good approximation. For instance, the difference in area, perimeter, or diagonal lengths relative to the true ellipse could be minimised [18]. However, in the first instance it seems worthwhile to constrain the solution to tangent continuous constructions. If the ellipse to be approximated has semi-major and semi-minor axes of length a and b then in the simplest scheme the approximation consists of four circular arcs with centres $(\pm h, 0)$ and $(0, \pm k)$ and radii $a - h$ and $b +$

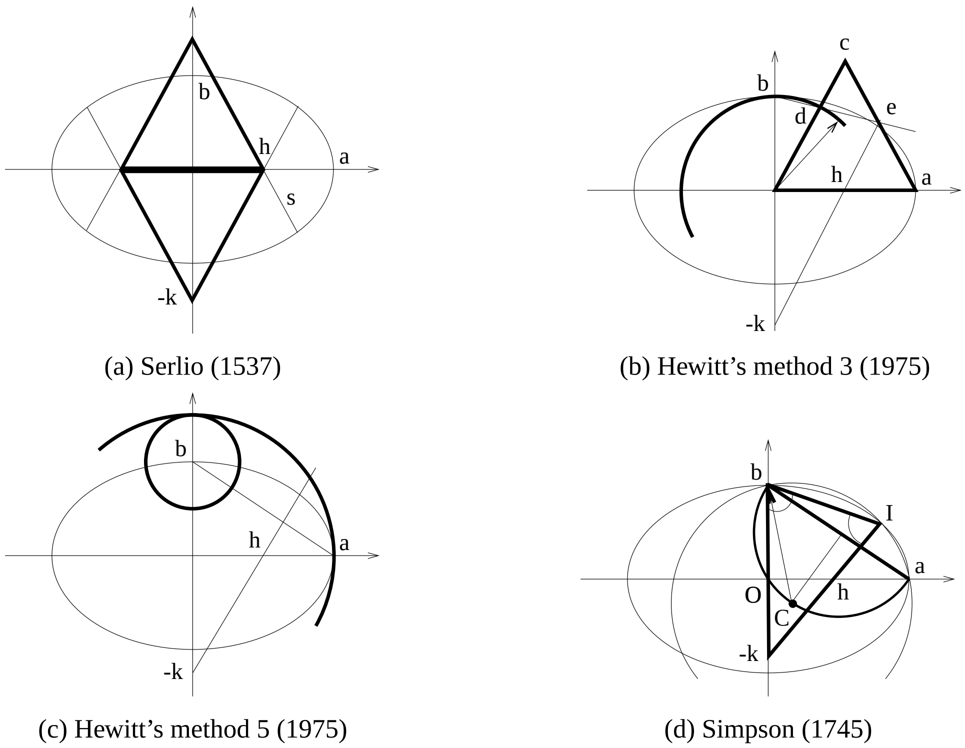
[a] As an example, TCAM Development's TwinCAD package cannot generate true parallels to an ellipse, but instead approximate them either using projected circles or polylines.

k respectively which pass through the extremal points of the ellipse. To ensure tangent continuity the arcs' centres must lie on the common normal to the joint and the geometry will be as shown in figure 1. The radii are the lines $\overline{kj}$ and $\overline{hj}$ and have lengths

$$\begin{aligned} \overline{hj} &= a - h \\ \overline{hj} &= \overline{kj} - \overline{kh} = (b + k) - \sqrt{h^2 + k^2} \end{aligned}$$

which leads to the constraint

$$h = \frac{k - \frac{a-b}{2}}{\frac{k}{a-b} - 1}.$$



(a) Serlio (1537)

(b) Hewitt's method 3 (1975)

(c) Hewitt's method 5 (1975)

(d) Simpson (1745)

Fig. 2. Diagrams illustrating various four-centred arc construction procedures.

Computing the optimal construction can be computationally expensive as it is often not possible in closed form. For instance Rosin [19] performed a one dimensional search over a range of values of h to find the best four-arc in terms of various error norms; i.e. $L_1 = \sum_i e_i$, $L_2 = \sum_i e_i^2$, and $L_\infty = \max_i e_i$, where e_i is an approximation to the distance from a point p_i on the arc along the normal to the ellipse, and p_i are uniformly sampled along the four-arcs.

More recently, for the case of the L_∞ norm Qian and Qian [20] provided a more efficient solution. They provide an analytic dimensionless function of the optimal $\frac{h}{a}$ versus

Table 1. Simple fractional factors for h and k. Included are the aspect ratios at which the construction minimises the maximum error.

h	$11/10$	$10/9$	$9/8$	$8/7$	$7/6$	$6/5$	$11/9$	$5/4$	$9/7$	$13/10$	$4/3$
k	6	$11/2$	5	$9/2$	4	$7/2$	$13/4$	3	$11/4$	$8/3$	$5/2$
a/b	7.71	6.91	6.11	5.32	4.53	3.75	3.36	2.97	2.58	2.45	2.20

h	$11/8$	$7/5$	$10/7$	$13/9$	$3/2$	$14/9$	$11/7$	$8/5$	$13/8$	$5/3$	$17/10$
k	$7/3$	$9/4$	$13/6$	$17/8$	2	$19/10$	$15/8$	$11/6$	$9/5$	$7/4$	$12/7$
a/b	1.94	1.82	1.69	1.63	1.44	1.29	–	–	–	–	–

$\frac{b}{a}$ in implicit form. Its solution still requires a one dimensional search, although a good explicit approximation to the implicit function can be made which enables a direct, fast solution.

For any particular ellipse there is a range of possible solutions to this equation and the literature includes various construction methods which provide particular solutions [19, 21]. Diagrams illustrating the construction of several typical examples are shown in figure 2. Further details can be found in [19, 21].

In all cases the methods fix h and k to some factors of $(a - b)$, and can be split into three categories. The first uses constant rational factors and is exemplified by French's method with factors $\left\{\frac{3}{2}, 2\right\}$. Next, are methods in which the factors are still constants, but are now irrational. Two such constructions are Hewitt's method 3 and the construction mentioned in Gridgeman [18]. Their factors are $\left\{\frac{1}{\sqrt{3}-1}, \frac{\sqrt{3}}{\sqrt{3}-1}\right\}$ and $\left\{\frac{1}{1+\sqrt{2}}, \frac{1}{1+\sqrt{2}}\right\}$ respectively. Finally, the remainder such as Hewitt's method 5 and Simpson's construction produce factors which are functions of a and b. These are $\left\{\frac{a+b+\sqrt{a^2+b^2}}{2a}, \frac{a+b+\sqrt{a^2+b^2}}{2b}\right\}$ and $\left\{\frac{a+b+\sqrt{a^2+6ab+b^2}}{a-b+\sqrt{a^2+6ab+b^2}}, \frac{a+3b+\sqrt{a^2+6ab+b^2}}{4b}\right\}$ respectively.

If the approximation errors are calculated [19] then we see that for most constructions the error monotonically increases with increasing aspect ratio (figure 3a). The exceptions are the first two constructions which have local optima. Investigating further we consider additional constant factors of $(a-b)$. If $k = f \times (a-b)$ then maintaining tangent continuity requires $h = \frac{2f-1}{2f-2}$. This enables all the simple values of h and k using just small fractional values to be generated as shown in table 1. It should be noted that the constructions do not work for all aspect ratios. That is, if the desired aspect ratio is too large then the circles centred at $(0, \pm k)$ touch the other two circles on the wrong side of the *X* axis, and so the ellipse approximation is not formed correctly. Thus the breakdown point can be determined as occurring when the circles centred at $(0, \pm k)$ intersect the *X* axis on the inside of the ellipse rather than the outside, which is at $\frac{a}{b} = 2f - 1$.

On plotting the errors a distinct pattern can be seen (figure 3b). The range of factors produces a family of solutions from which each member provides a close to optimal approximation at one aspect ratio.[b] Outwith that point the errors increase rapidly to substantial values that are particularly noticeable at low aspect ratios. Such behaviour is in

[b] The optimal approximation (described in more detail in [19]) is calculated numerically, and relates to the fit that minimises the maximum Euclidean distance between the ellipse and the tangent continuous four-arc construction.

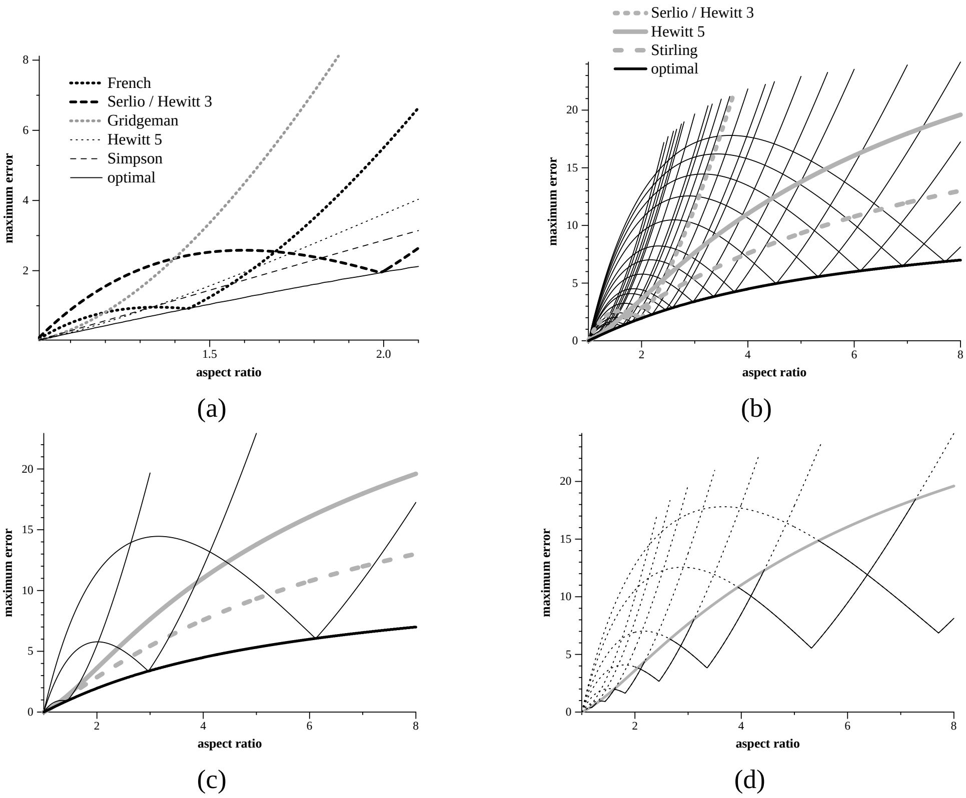


Fig. 3. a) Maximum error in the constructions producing tangent continuous ellipse approximation. b) Maximum error in the ellipse approximation as a function of aspect ratio. The thin black lines show errors from the family of constructions listed in table 1. A characteristic pattern is evident: in addition to the expected zero error for a circle most constructions have another local optimum. This is located at increasing aspect ratios as k increases (h decreases). c) Approximation errors of $\left\{\frac{9}{8},5\right\}$, $\left\{\frac{5}{4},3\right\}$, and $\left\{\frac{3}{2},2\right\}$ factors. d) Approximation errors of every third factor ensures a more accurate approximation than Hewitt's method 5.

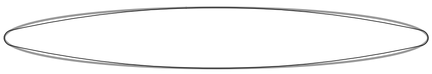
contrast to many other methods such as Simpson's and Hewitt's method 5 in which the error increases monotonically with increasing aspect ratio.

Instances of the $(a-b)$ factor approach cannot compete against Hewitt's method 5 in terms of consistently good performance. Nevertheless it does mean that constructions providing accurate approximations can be achieved by selecting the appropriate factor to suit the desired aspect ratio, although this requires considerable computing power. However, even in the Renaissance it would have been feasible to experimentally test various factors and aspect ratios and derive a few simple[c] values and their range of applicability. For instance, a good choice would be the three simple sets of factors $\left\{\frac{9}{8},5\right\}$, $\left\{\frac{5}{4},3\right\}$, and $\left\{\frac{3}{2},2\right\}$ which produce constructions tuned to aspect ratios 6.11, 2.97, and 1.44. As can be seen from figures 3c and 4, selecting the appropriate construction gives for the most part

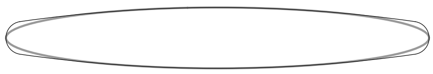
[c] Since the denominators of these factors are all powers of two they are especially straightforward to determine by geometric constructions as a length can be repeatedly bisected using a compass and straight edge.

more accurate approximations than Hewitt's method 5, which is the most accurate straightforward construction we know of.[d] Alternatively, selecting every third set of factors from table 1 generates a set of four-arcs that ensure a more accurate approximation than Hewitt's method 5 as shown in figure 3d.

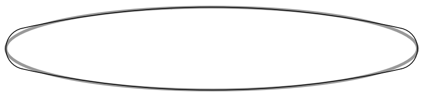
It is interesting to note that Gridgeman's construction which is the special case for $h = k$ can be considered as a limiting case in which the local optimum coincides with the global optimum at $\frac{a}{b} = 1$. In other words, this construction is particularly poor since it only works well for near circular ellipses.



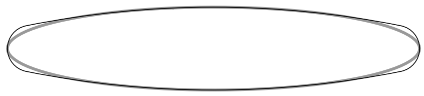
(a) $\left\{\frac{9}{8}, 5\right\}(a-b)$



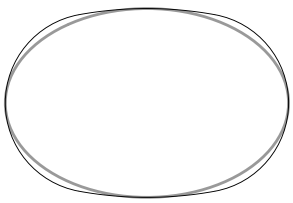
(b) Hewitt 5



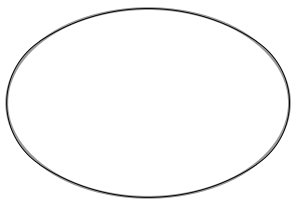
(c) $\left\{\frac{9}{8}, 5\right\}(a-b)$



(d) Hewitt 5



(e) $\left\{\frac{9}{8}, 5\right\}(a-b)$



(f) Hewitt 5

Fig. 4. Approximation of ellipses with (a) & (b) $\frac{a}{b} = 7$, (c) & (d) $\frac{a}{b} = 5$, (e) & (f) $\frac{a}{b} = 1.5$. Even outwith its optimal aspect ratio the $\left\{\frac{9}{8}, 5\right\}(a-b)$ factor method does well compared to Hewitt's method 5. However, eventually the errors incurred by the factor method become significant, and it is clearly outperformed by Hewitt's method 5 in the last example.

3. Conclusions

In many areas such as CAD, architecture, engineering, construction, and computer vision there is a need to approximate ellipses by a simpler representation such as circular arcs. With modern computing power it is possible to apply iterative numerical methods to determine the optimal approximation. Nevertheless, for many applications it is sufficient and more convenient to be able to use a simple and direct method. Stirling's oval is one of the best of the closed form solutions, producing a consistently accurate approximation.

[d]Although Simpson's construction gives better approximations it is not useful in practise since it requires the ellipse to be drawn first to guide the approximation! Of course, although the geometric construction is limited the algebraic form is still useful as providing an accurate approximation.

The advantage of the $(a - b)$ factor method described in this paper is that it is extremely straightforward. Selecting just three simple pairs of factors (corresponding to compact, medium, and elongated ellipses) enables accurate approximations to be generated with the minimum of effort.

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