On Serlio's constructions of ovals

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Abstract

In his celebrated *Tutte l'Opere d'Architettura* published over the period 1537–1575 Sebastiano Serlio introduced four techniques for constructing ovals which have thereafter been applied by many architects across Europe. Using various geometric forms (i.e. the triangle, square, and circle) as a basis they produced ovals made up from four circular arcs. This paper analyses both Serlio's constructions and some of the many possible alternatives and evaluates their accuracy in terms of the ovals' approximations to an ellipse.

1 Introduction

After the dearth of the Middle Ages, when architectural principles were kept secret by the guilds, the Renaissance was witness to an explosion of architectural treatises and handbooks [11]. These were mostly written by architects, in manuscript form until the end of the fifteenth century, and provided a combination of theories, rules and patterns concerning all aspects of architecture. Contents range over the suitability and preparation of building materials, the design of plans, facades, and ornamentation, theories of beauty, and primers on geometry, to details of suitable inscriptions for tombs, and methods for repelling and exterminating insects.

The main prior work was Vitruvius's *De Architectura* which was (and remains) the only extant architectural treatise from Roman times. Written before AD 27 it has, despite its failings, enjoyed substantial popularity; numerous manuscript copies can be traced through Europe up until the Renaissance [9] and more than twenty copies made in the fifteenth century alone are known [5]. According to Vitruvius the design of a harmonious, pleasing building requires careful attention to the proportions of its parts. Their sizes are simple multiples of a unit length called the module. The first treatise of the Renaissance (written around 1450 and published in 1485, thirteen years after his death) was Alberti's *De Re Aedificatoria* and it followed (or at least was inspired by) classical buildings, and of course Vitruvius. Alberti recommended nine basic geometric shapes. First there are the regular polygons: the square, hexagon, octagon, decagon, and dodecagon, all of which can be constructed from a circle (which he considered the ideal form [32]). In addition, there are three rectangles based on extensions of the square. Buildings could then be planned by taking these forms as the underlying components, and combining them to form a pleasing plan.

Following Alberti came a host of other authors elaborating, recodifying, and extending the previous writings.[1] During this period the geometry of the circle predominated, holding "an almost magical power" over the architects [32]. In contrast, the ellipse "was as emphatically rejected by High Renaissance art as it was cherished in Mannerism" [22]. Not only was the circle considered to have the most perfect and beautiful form, but its preeminence over the ellipse was bolstered by arguments based on the circular movements of the human body [22].[2]

Eventually the fashion shifted from Alberti's restrictive circle geometry, and the ellipse came to provide a "convenient form for the Baroque spirit for dynamic expression" [19], not only for aesthetics, but also astronomy, mechanics, etc. For instance, somewhat later William Hogarth, in *The Analysis of Beauty*,

[1] For example, in the fifteen century we have amongst others Filarete, Georgio, and Grapaldi, while those of the next century include Vignola, Palladio, and Scamozzi. As can be seen, the majority of activity took place within Italy.

[2] Actually, the same anthropomorphic argument can be applied to ellipses [12]. As seen from above, the rotation of the two arms sweeps out a pair of circular arcs reminiscent of the circles used in one of Serlio's schemes to approximate ellipses.

states "...the oval has a noble simplicity in it, more equal in its variety than any other object in nature[3] ...[and] is as much to be preferred to the circle, as the triangle to the square[4]" An early proponent was Michelangelo, who considered the ellipse during his first project for the tomb of Julius II[5], and later used it in his design for the Piazza del Campidoglio.[6] Another pioneer in the use of the ellipse was Baldassare Peruzzi, who had a predilection for difficult or eccentric problems. However, he neither built nor published any related material, possibly due to his early death. It was left to his pupil, Sebastiano Serlio, to write the celebrated *Tutte l'Opere d'Architettura* which was published over the period 1537–1575. The work incorporated and acknowledged material left by Peruzzi, although the accusations of wholesale plagiarism by many (e.g. Vasari and Cellini) appear unfair. Serlio's approach to the writing of the architectural treatise was significantly new on several counts: [5, 11]

- For the first time a coordinated scheme of architectural education was devised.
- It contains the earliest detailed work on the five Orders. Serlio attempted to sort out the discrepancies between Vitruvius and actual Roman monuments, and establish clear consistent rules for the Orders.
- It provided practical design patterns, and was written for the use of architects rather than perusal by wealthy patrons and noblemen.
- It was written in Italian whereas most of the previous writings had been made in Latin.
- The illustrations were a substantial component of the work. In contrast to Alberti's original which contained none, while Vitruvius' had been long lost, out of the 155 pages of Serlio's third book 118 had one of more illustrations.

This combination of practicality and ease of use made it one of the most important architectural treatises, and numerous editions and translations were made. As Summerson said: "The books became the architectural bible of the civilised world. The Italians used them, the French owed nearly everything to Serlio and his books, the Germans and Flemings based their own books on his, the Elizabethans cribbed from him...". Of particular relevance to this paper, Serlio agreed with Alberti on the ascendance of the circular form, but extended the range to include amongst the polygons the Greek Cross and the oval, thus heralding a new approach to ecclesiastical architecture [18, 32]. Amongst the church patterns he included one with an elliptical plan, although he himself did not build any elliptical churches. Consequently, in large part encouraged by Serlio's book, many elliptical churches were built in Italy and Spain from the sixteenth century onwards.[7] These are covered in great detail by Wolfgang Lotz [18] and Vincenzo Fasolo [6]; see also [14, 19, 31]. Moreover, elliptical theatre design became popular from the end of the seventeenth century, first in Italy and then in France, while the ellipse also featured prominently in art [24]. Over the years the ellipse has continued to figure in architectural design. For example, instances in recent buildings are the striking "Lipstick Building" by Philip Johnson and John Burgee which consists of a stack of elliptical cylinders, the Tokyo International Forum designed by Rafael Viñoly which contains the vast elliptical Glass Hall, and the additions to the British Library's Reading Room by Foster and Partners (see figure 1). We also note that the ellipse occurs over a wide range of scales. Figure 2 shows examples, starting with small scale decorative details such as the patera, continuing to the scale of a single room (Robert Adam's hallway at Culzean Castle), and continuing beyond the scale of a single house to John Wood's conglomeration of the Royal Crescent at Bath.

In order to build elliptical structures a means of constructing the shape of an ellipse is required. The instructions found in books on building construction or technical drawing fall into three classes – geometric rules for drawing true ellipses, mechanical devices for drawing true ellipses, and geometric rules for drawing approximations to ellipses using four or more circular arcs. The methods are generally based on the geometric properties of ellipses which have been available from various classical treatises of conics. The most complete one that has survived, by Apollonius, was republished in Venice in 1541. However, this is predated by works such as Werner's conics (1522) and Dürer's geometry (1525). In fact, Sinisgalli asserts that Leonardo da Vinci was the first among the moderns to draw an ellipse, in around 1510. Thus there

[3] In chapter 12 - "Of Light and Shade".

[4] In chapter 4 - "Of Simplicity, or Distinctness".

[5] This view is disputed on lack of evidence [30].

[6] We note that there are three major architectural works in Rome in which the ellipse figures prominently: the Piazza del Campidoglio, the Colosseum, and Bernini's colonnade outside St. Peter's cathedral.

[7] For instance, some of the sixteenth and seventeenth century architects who designed elliptical churches are: Vignola (in 1550 this was the first built on an oval plan), Mascherino, Volterra, Vitozzi, Bernini, Rainaldi, and Borromini.

Figure 1: Examples of modern elliptical buildings

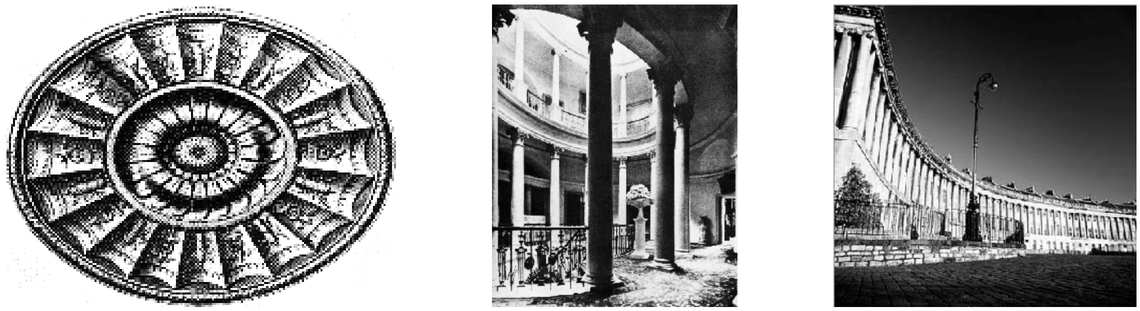


Figure 2: The ellipse in architecture at different scales

was plenty of theory available during the Renaissance. However, translating this into the practicalities of architecture could be problematic.

From the first class of ellipse constructions the best known is the “gardener’s method” in which two pins or pegs are placed at the foci and a length of string is attached to the pegs. The pen is pulled out against the pegs against the string and around, thereby drawing out the ellipse. While adequate for drawing rough, small ellipses there are several difficulties in scaling the approach up to tackle accurate, large ellipses. First, in order to draw the full $360°$ properly the string must be looped rather than tied to the pegs. Even so, the knot in the string can interfere with the drawing process. Second, particularly with long sections of string it is difficult to maintain a constant tension, and even variations in humidity and temperature can have an effect. And finally, if the pen or marker is not kept perfectly perpendicular to the drawing surface then additional inaccuracies will occur.

From the next class of ellipse constructions, drawing devices such as trammels were used in the Renaissance, and there were many contemporary developers of ellipse compasses – including well known artists such as Leonardo da Vinci, Dürer, and possibly Michelangelo [15, 25]. These enabled reasonably precise ellipses to be drawn in plans, but again could not be readily applied to the large scale marking out of buildings.

This leaves the third class of ellipse constructions which, at the expense of only approximating the shape of the ellipse, provides a more practical tool for the architect and builder. In fact, there is a double benefit since the continually varying curvature of the ellipse otherwise complicates matters. For instance, measuring the perimeter along a section of an ellipse can only be approximated; there is no simple analytical solution. Also, for small ellipses such as door or window arches, precise building construction would require a large range of brick shapes [20]. As in indication of the difficulty in dealing with the ellipse, Batty Langley [16] in an eighteenth century builder’s manual reckons on an extra 50% expense in workmanship and materials for constructing elliptical walls as compared to straight walls. Although there are various possible approaches to approximating the ellipse (e.g. Blackwell describes a polygonal approximation [2]) the most popular approach has been to use a combination of circular arcs. The advantage of working with circles is they provide a powerful geometric tool: with circles alone one can copy angles, bisect angles and lines, construct perpendiculars to lines, construct equilateral triangles, etc. [4]. Moreover, they keep the construction process simple and reliable, and provide a reasonably accurate oval with a pleasing appearance. In his unpublished papers Peruzzi roughly sketched out two such constructions (types III and IV described below);[8] see figure 3. In his *Tutte l’Opere d’Architettura* Serlio provides, in addition to a plan of an elliptical

[8]It is perhaps not coincidental that these scribbles are adjacent to his design for Palazzo Massimo, in which he uses a curved wall

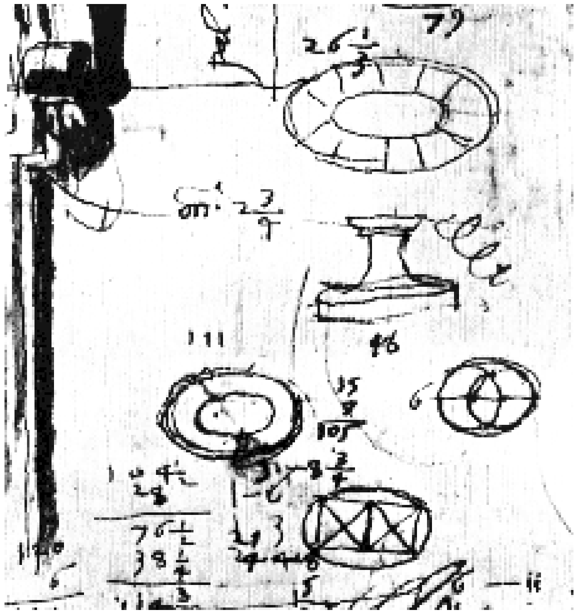


Figure 3: Peruzzi's sketch containing two oval constructions

church and a method for constructing true ellipses, four construction methods for drawing ovals, namely piecewise circular approximations to the ellipse.

2 Serlio's Four Oval Constructions

Serlio's four alternative constructions for approximating the ellipse from his First Book are shown in figure 4. Each generates an oval consisting of four circular arcs with centres $(\pm h, 0)$ and $(0, \pm k)$ and radii $a - h$ and $b + k$ respectively. Constructions II-IV have been called by Giulio Troili in the seventeenth century and others since by the more descriptive terms *ovato longo*, *ovato*, and *ovato tondo*. More recently, the general term *quadrarc* has been used in their description and analysis [8, 27]. Kitao [13] incorrectly states that constructions II–IV are all special cases of construction I. In fact, the more general construction would be one in which an arbitrary type of triangle is used with its sides extended by an arbitrary length. This would give many possible alternative constructions for any desired ellipse (i.e. any semi-major and semi-minor axes a and b). The important factor is that due to the underlying triangle all the constructions exhibit circular arcs with tangent continuity.[9] This geometric constraint can be algebraically expressed as

$$h = \frac{k - \frac{a-b}{2}}{\frac{k}{a-b} - 1}.$$

Serlio's constructions I and IV contain an underlying equilateral triangle. Both constructions II and III use an isosceles triangle made up from half a square, but extended by different amounts, $\frac{w}{\sqrt{2}}$ and w respectively, where w is the length of the square's side. In addition, the three square construction described later is made up of a similar triangle, but only extended by $\frac{w}{2}$.

The presence of circles, triangles, squares, etc. underlying the constructions is not surprising since the combination of geometric primitives was a favourite device of artists and architects in the Renaissance (as well as in earlier and later times). In Italy the normal proportions were based on the equilateral triangle, leading to the geometric system *ad triangulum* [9]. Many examples of a preoccupation with underlying geometric forms are found in numerology ("sacred geometry"), leading on to architecture, and as such are particularly evident in important buildings such as churches.

to cleverly combine two adjacent existing sites into a single building.

[9] Despite Arnheim's assertion to the contrary [1], circular arcs *can* be joined easily and pleasingly as demonstrated in these oval constructions.

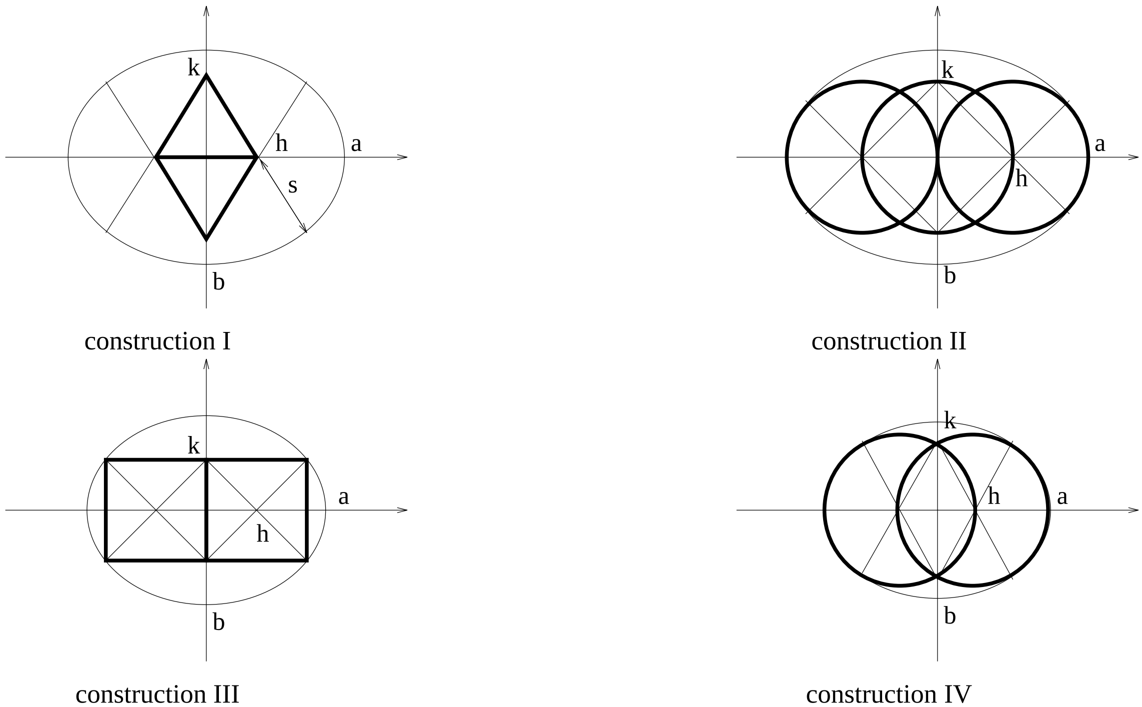


Figure 4: Serlio's four constructions for drawing ovals

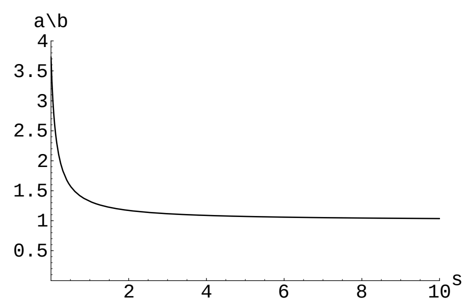


Figure 5: Aspect ratio of Serlio's construction I

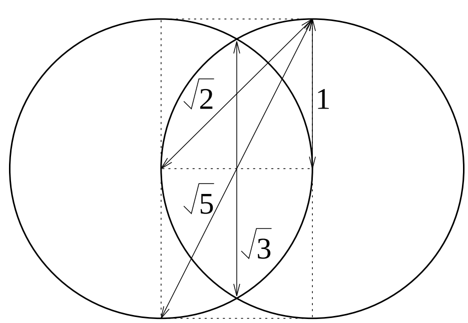


Figure 6: The trinity of roots contained in the Vesica Piscis

2.1 Construction I

This is the only one of Serlio's constructions that enables ovals with varied eccentricities to be drawn. It consists of two equilateral triangles whose bases are centred on the origin. Their intersections with the axes determine h and k, which can be expressed as

$$h = \frac{a-b}{\sqrt{3}-1}; \;\; k = \frac{\sqrt{3}(a-b)}{\sqrt{3}-1}.$$

The lengths of the circular arcs are specified by extending the triangles' diagonal sides by a length s. When s is increased both a and b are increased, and so it is not straightforward to (geometrically) choose correct values of h and s so as to achieve specified values of a and b. In fact, the ratio is given by

$$\frac{a}{b} = \frac{h+s}{h(2-\sqrt{3})+s}.$$

Figure 5 shows how, for a triangle with unit length sides, increasing s from zero initially rapidly increases the circularity of the oval, while as s increases beyond one the aspect ratio very slowly decreases, reaching a perfect circle in the limit. Thus not all aspect ratios can be achieved with this method since two arcs per quadrant are only drawn when $s > 0$ such that $\frac{a}{b} \leq \frac{1}{2-\sqrt{3}} \approx 3.732$. The ratio of the radii of the two circular arcs also varies, and can be shown to be $\frac{s}{2h+s}$.

2.2 Construction II

The second construction uses three circles, each passing through their neighbouring circle(s) centres. The centres of the circular arcs are at

$$h = k = \frac{a}{2}$$

and produce an approximate ellipse with the fixed aspect ratio of $\frac{a}{b} = \sqrt{2} \approx 1.414$. The ratio of the radii of the arcs is $\sqrt{2}-1$. Serlio considered this oval to be very similar to the shape of a natural egg. To verify this a rough test was made in which the typical aspect ratios of the eggs of one hundred randomly selected species of European bird were obtained [23]. The mean aspect ratio was found to be 1.38 (with standard deviation .09) which is indeed closer to construction II than the constructions III or IV. We also note that such an arrangement of circles was often used by artists in their designs. An extensive example of its use is given by Pinturicchio's frescos in the Library of Siena Cathedral [3].

2.3 Construction III

The third construction uses yet another geometric primitive as its basis: the square. This yields

$$h = k = \frac{a}{\sqrt{2}+1}$$

with a fixed aspect ratio of $\frac{a}{b} = \frac{\sqrt{2}+1}{2\sqrt{2}-1} \approx 1.320$, and the ratio of the radii of the arcs is $\frac{1}{2}$.

2.4 Construction IV

The final construction was recommended by Serlio for its beauty, and is the simplest and quickest to construct. Not surprisingly, this was the standard ellipse approximation used in architectural practise [13]. The configuration of two intersecting circles is common in sacred geometry [17] where it is called the *Vesica Piscis*, its symbolism often appearing in Egyptian, Christian, etc iconography. As an example of its numerical significance, its proportions contain the three basic roots, as shown in figure 6. Kitao also notes the simple ratios present in the relationship of its parts:

1. the lengths of the arcs are 1:1,

2. the radii of the arcs are 1:2,

3. the major axis of the oval to the smaller arc radius is 1:3.

Like construction I there is an underlying equilateral triangle, although this is not required to be explicitly drawn to locate the arcs. The centres of these arcs are located at

$$h = \frac{a}{3}; \;\; k = \frac{a}{\sqrt{3}},$$

and its fixed aspect ratio is $\frac{a}{b} = \frac{3}{4-\sqrt{3}} \approx 1.323$. Thus this construction provides a very similar approximation to the previous construction.

3 Other Oval Constructions

Having run through Serlio's four oval constructions we now consider alternative approaches. Some are variations on the methods already described while others are extensions or improvements.

3.1 Three square construction

In the same way that Serlio's construction II increased the two circles in construction IV to three circles we consider increasing the two squares in construction III to three squares (figure 7a). The same shaped underlying isosceles triangle remains, but is extended by a shorter section, giving

$$h = k = \frac{2a}{\sqrt{2}+2}.$$

The fixed aspect ratio is $\frac{a}{b} = \frac{1+\sqrt{2}}{3-\sqrt{2}} \approx 1.522$, and the ratio of the radii of the arcs is $\frac{1}{3-\sqrt{2}} \approx 0.631$.

3.2 Four circle construction

Continuing Serlio's constructions of two and three intersecting circles we show four intersecting circles in figure 7b, giving

$$h = \frac{3}{5}a; \;\; k = \frac{\sqrt{3}}{5}a.$$

with the fixed aspect ratio is $\frac{a}{b} = \frac{5}{\sqrt{3}+2} \approx 1.340$ and the ratio of the radii of the arcs is $\frac{1}{\sqrt{3}+1} \approx 0.366$.

Increasing from two to three circles produced an increase in aspect ratio, but moving from three to four has decreased it almost back to the two circle aspect ratio. This is due to the alternation of the position of k at the top of the central circle or at the intersection of the middle two circles depending on whether there are an odd or even number of circles used.

3.3 Half-square triangle

Just as Serlio's construction I extends an equilateral triangle by variable amounts to generate ovals over a range of eccentricities we apply the same technique to the isosceles triangle underlying his constructions II and III, yielding

$$h = k = \frac{a-b}{2-\sqrt{2}}$$

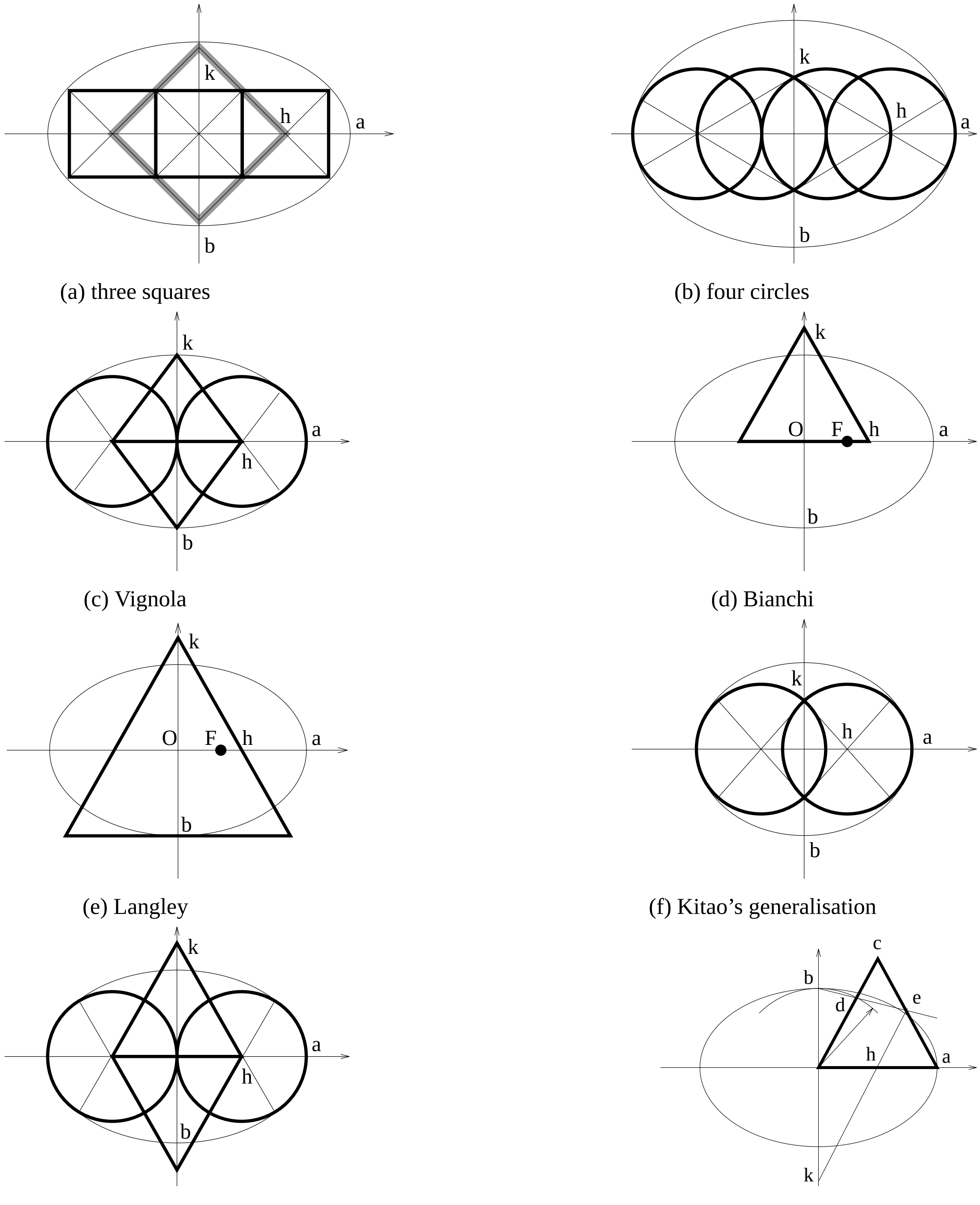
with an aspect ratio of $\frac{a}{b} = \frac{h+s}{(\sqrt{2}-1)h+s}$ and the ratio of the radii of the arcs $\frac{s}{\sqrt{2}h+s}$. A similar behaviour of aspect ratio to extension length is observed: for small values of s minor changes cause large differences in aspect ratio.

3.4 Vignola's construction

Kitao [13] describes a construction by Giacomo Vignola based around the Pythagorean triangle with side lengths 3, 4, and 5 as shown in figure 7c,

$$h = \frac{a}{2}; \;\; k = b = \frac{2}{3}a$$

giving an fixed aspect ratio $\frac{a}{b} = \frac{3}{2}$ with the ratio of the radii of the arcs as $\frac{3}{8}$. Golvin also claims this approach was used by the Romans in their building of ampitheatres [7].



(a) three squares (b) four circles (c) Vignola (d) Bianchi (e) Langley (f) Kitao’s generalisation (g) Mott (h) Hewitt’s method 3

Figure 7: Alternative oval constructions

3.5 Kitao

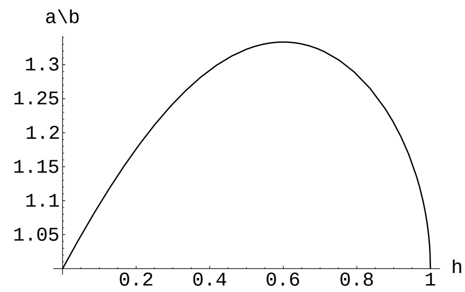
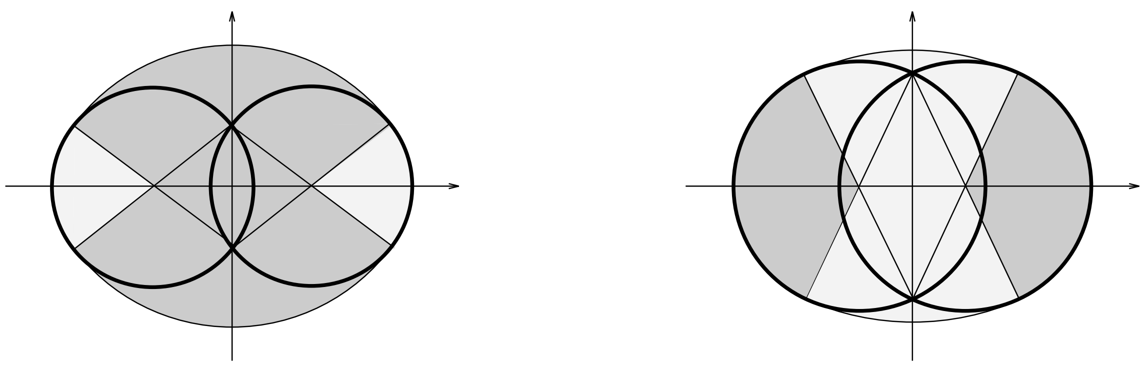


Figure 8: Aspect ratio of Kitao's construction (unit circle radius)



(a) (b)

Figure 9: Two ovals with identical aspect ratios generated using Kitao's construction

Kitao [13] generalises Serlio's construction IV by allowing the circles to move closer or further apart (figure 7f), resulting in

$$h = \frac{3a - 2b \pm \sqrt{4ab - 3a^2}}{4}; \;\; k = \sqrt{a(a - 2h)}.$$

The solution becomes complex when $\frac{a}{b} > \frac{4}{3}$. On examination we see the cause for the breakdown at this point in the unusual behaviour of the oval's aspect ratio as the underlying circles are moved apart. Starting in the limit from concentric circles which produce a circle, the aspect ratio reaches a peak of $\frac{a}{b} = \frac{4}{3}$ when $h = \frac{3}{5}r$ (where r is the radius of the underlying circles) and thereafter decreases to unity again when the circles no longer overlap – see figure 8. Thus, for any aspect ratio achievable by Kitao's method there are two possible constructions. Curiously enough, the ratio of the radii of the arcs is constant, namely $\frac{1}{2}$ – the only such case of a fixed ratio of arc radii over all the variable aspect ratio oval constructions analysed in this paper. Despite having identical aspect ratios the ovals differ, as demonstrated in figure 9. Compared to the oval in (a), the oval in (b) is made up with arcs of larger radii, and therefore has a squarer appearance.

3.6 Bianchi's construction

Although Serlio's construction I does allow a range of oval eccentricities to be approximated it does not easily enable any specific eccentricity to be determined since the two parameters (triangle length and extension) interact. Later constructions overcame this difficulty. For instance, Paolo Bianchi's *Instituzione Practica dell' Architettura Civile* in 1766 provides the following scheme (see figure 7d):

1. mark point F such that the length $Fa = Ob$
2. point h is marked such that $Oh = \frac{4}{3}OF$
3. an equilateral triangle is constructed, its vertices defining h and k:

$$h = \frac{4}{3}(a - b); \;\; k = \frac{4}{\sqrt{3}}(a - b).$$

This approach operates up until the breakdown aspect ratio of $\frac{a}{b} > \frac{8}{\sqrt{3}} - 1 \approx 3.619$.

It is interesting to note that Bianchi's approach provides identical ovals, but using an alternative construction, to those of the Slantz method [33] described in current engineering drawing books. Another variation producing the same oval was given by Langley [16] in which the equilateral triangle is drawn such that it touches the bottom of the ellipse (figure 7e).

3.7 Equilateral triangle constructions

We note that many oval constructions involve the use of equilateral triangles. Serlio's constructions I and IV and Bianchi's construction have already been described. Two further more modern examples are shown. Mott's construction (his method number 7.8) [21] also involves two circles like Serlio's construction IV. However, the circles touch rather than intersect, and an equilateral triangle is drawn with its bottom corners in the circle centres (figure 7g), yielding

$$h = \frac{a}{2}; \;\; k = \frac{\sqrt{3}}{2} a.$$

Although it is similar to Vignola's construction with the Pythagorean triangle replaced by the equilateral triangle there is a further difference: the radius of the arc centred at k is set to $k + b$; this enables a range of aspect ratios (although only $[1 + \frac{1}{\sqrt{3}}, \frac{\sqrt{7}+\sqrt{3}}{2}] \approx [1.5774, 2.1889]$) to be constructed, but the ovals are no longer tangent continuous.

Hewitt's method 3 [10] is somewhat different in that unlike the previous examples the equilateral triangle is centred on the semimajor axis rather than the oval centre (figure 7h). The procedure is:

1. the arc centred at the origin with radius b is drawn out; it cuts the triangle at d
2. the straight line through b and d is drawn, and cuts the triangle again at e
3. the line through e parallel to the triangle side Oc intersects the axes at h and k, yielding

$$h = \frac{(a-b)}{\sqrt{3}-1}; \;\; k = \frac{\sqrt{3}(a-b)}{\sqrt{3}-1}.$$

Thus it provides precisely the same oval as Serlio's construction I. Its advantage is that given the desired aspect ratio of the oval then the construction follows easily. Serlio's method provides no means of achieving a specific aspect ratio except by trial and error variation of h and s.

3.8 Robert Simpson's construction

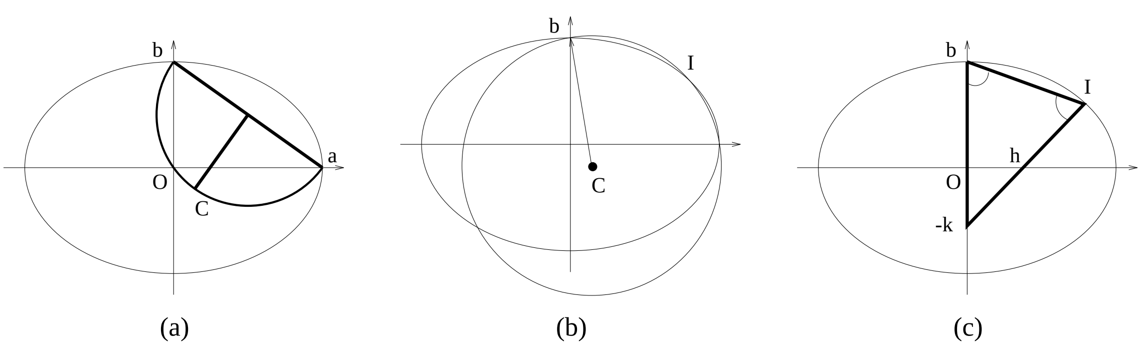


Figure 10: Simpson's oval construction

In around 1744, motivated by his interest in astronomy and the orbits of the planets, James Stirling devised an approximation in which not only do the arc joins have tangent continuity, but they also lie on the true ellipse [29]:

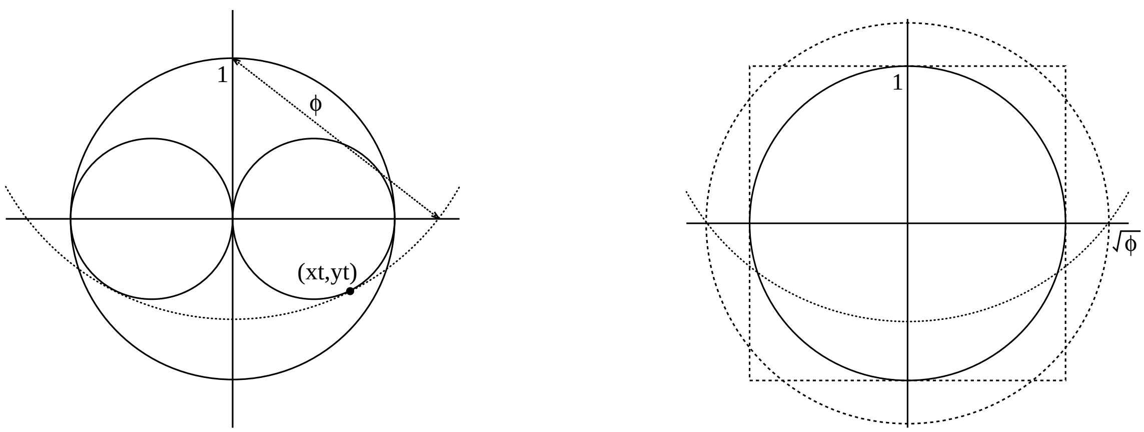
$$h = \frac{(a-b)\left(a+b+\sqrt{a^2+6ab+b^2}\right)}{a-b+\sqrt{a^2+6ab+b^2}}; \;\; k = \frac{(a-b)\left(a+3b+\sqrt{a^2+6ab+b^2}\right)}{4b}.$$

Stirling proposed to Robert Simpson that he find a geometric construction for his algebraic description. In a manuscript Simpson provided the following solution in 1745, shown in figure 10

1. the semicircle baO is drawn and bisected at point C
2. the circle[10] with centre C and radius Cb intersects the ellipse at point I
3. the isosceles triangle is drawn as shown; its intersection with the axes provides h and k

Stirling's approximation was recently shown to be extremely accurate [27]. Unfortunately however, due to the shallow intersection of the circle and ellipse in stage 2 it is difficult to determine point I, making its geometric construction rather impractical. In addition, it requires the true ellipse to be drawn prior to the oval!

3.9 Squaring of the circle



(a) (b)

Figure 11: Squaring of the circle

In the rather different context of sacred geometry Lawlor [17] describes a construction that could lead to an oval. Figure 11 illustrates an attempt at squaring the circle, i.e. using only a compass and straight-edge to construct a square with perimeter equal to the circumference of a given circle. First a circle with unit radius is drawn with two half-sized circles inside. Taking the top of the outer circle as centre, a circular arc is drawn such that it is tangent to both the inner circles as shown in figure 11a. The point of contact is

$$(x_t, y_t) = \frac{1}{\sqrt{5}}(\pm\phi, -1)$$

where $\phi = \frac{\sqrt{5}+1}{2} \approx 1.618$ is the Golden Ratio which often appears in sacred geometry and theories of proportions and aesthetics. The radius of the arc is thus ϕ, and the intersection of the arc with the X axis is at $x = \sqrt{\phi}$. Drawing another circle centred at the origin and passing through the intersection (i.e. with radius $\sqrt{\phi}$) produces a circle whose circumference is $2\pi\sqrt{\phi} \approx 7.99$. The perimeter of the square circumscribing the initial circle equals eight, and thus provides a very close estimate to the final circle's circumference. Figure 11a also provides a means of constructing a circular approximation to an ellipse, using the inner circles (radius $\frac{1}{2}$) and the arcs of radius $\sqrt{\phi}$ joining at (x_t, y_t). The centres are at

$$h = \frac{a}{2}; \;\; k = a.$$

the fixed aspect ratio of the oval is ϕ, and the ratio of the radii of the arcs is 2ϕ.

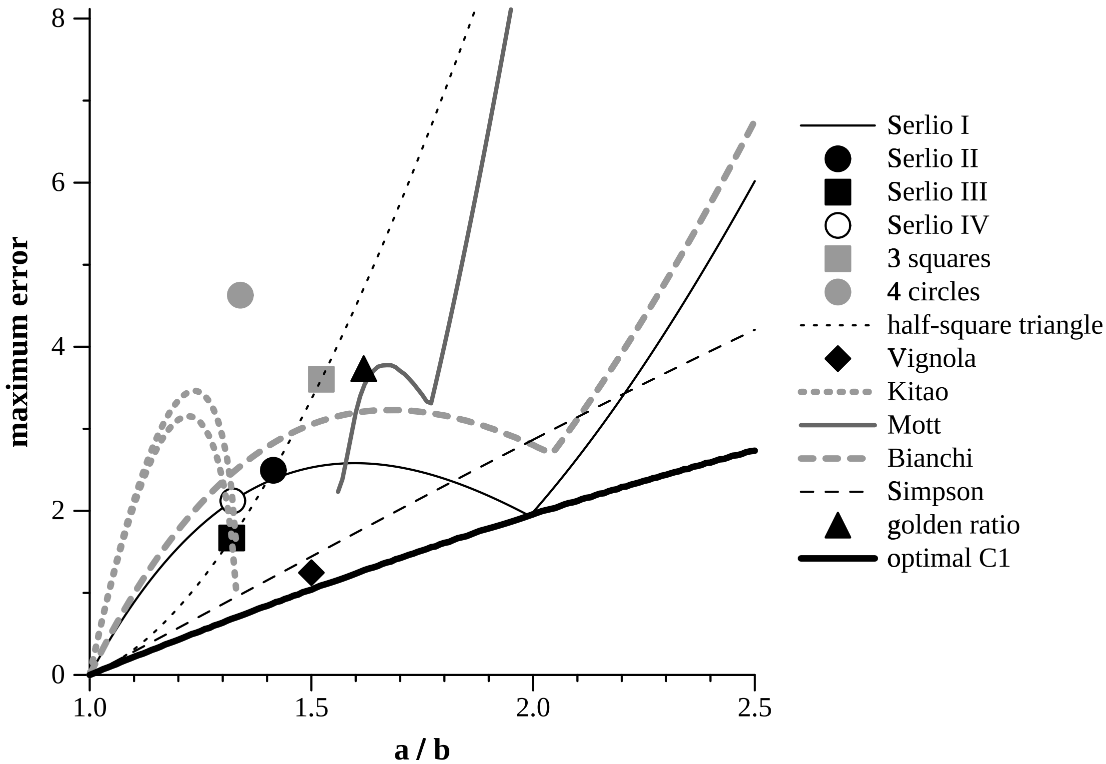


Figure 12: Approximation errors of the ovals with respect to the desired ellipse

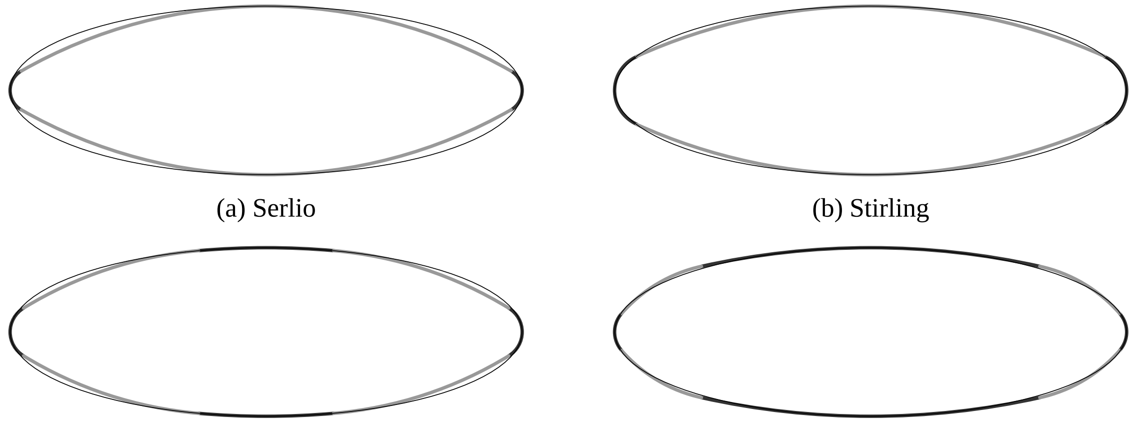
4 Fidelity to the Ellipse

Having defined a substantial range of oval constructions the obvious question is: how well do they approximate the desired ellipse? The analytic solutions are complicated by the involvement of the elliptic curve and so a numerical approximation to the approximation errors is calculated instead. The circular arcs are sampled at approximately equally spaced points, and at each point the distance along the normal to the ellipse is approximated. See [26, 27] for more details. Fixing $b = 100$ and using 1000 sample points in total the graph in figure 12 was generated. Although only the maximum error is displayed the average error exhibits a similar pattern.

The error incurred by the optimal tangent continuous (C^1) oval (which was numerically estimated by a 1D search) [27] is included for reference. We can see that Serlio's constructions do reasonably well, but are certainly not the closest to the ellipse (although of course this may not reflect their aesthetic qualities). For instance, Simpson's construction does uniformly well and is generally superior (it was previously found to outperform most of the more modern methods too [27]). In addition, Vignola's construction does especially well. Nevertheless, the extensions of all of Serlio's constructions (i.e. half-square triangle, 3 squares, 4 circles, and Kitao's generalisation) mostly perform poorly (the exception is the half-square triangle construction at low eccentricities). This shows how apparently plausible constructions do badly, and suggests that some care was taken in developing Serlio's original constructions. This argument is supported by the fact that Serlio's construction I is always better than Bianchi's, and mostly better than Mott's construction. It is interesting to note that for ovals with aspect ratios just less than two Serlio's construction I approaches the optimal approximation.

The oval approximations of ellipses with low eccentricity are mostly good, and the errors are barely noticeable. For more elongated ellipses we can see the discrepancies more clearly, as shown in figure 13. The first two show four-arc ovals discussed in this paper, and the superiority of Stirling's approximation over Serlio's construction is obvious. The following two ovals are constructed using eight-arcs – details are given in [28]. Naturally, using more arcs it is possible to improve the accuracy of the approximation to the ellipse, as is achieved by Walker's method. Perhaps surprisingly, a number of eight arc ovals were found to be inferior to four arc ovals, as demonstrated by Lockwood's oval; see also Rosin [27].

[10] This circle is in fact the locus of arc joints satisfying the C^1 constraint [27].



(a) Serlio (b) Stirling (c) Lockwood (d) Walker

Figure 13: Various oval constructions showing their discrepancies against the ellipse

5 An Application

We finish by showing that Serlio himself made good use of his methods of oval construction. Figure 14 shows his plan for an oval church from the *Tutte l'Opere d'Architettura*. The best fit ellipse is overlaid in figure 14a and the discrepancies at the diagonals are evident. The best fit oval was determined by performing an optimisation over all parameters using Powell's method, and as can be seen in figure 14b not only does it provide a better fit to the church's perimeter but is clearly Serlio's construction IV.

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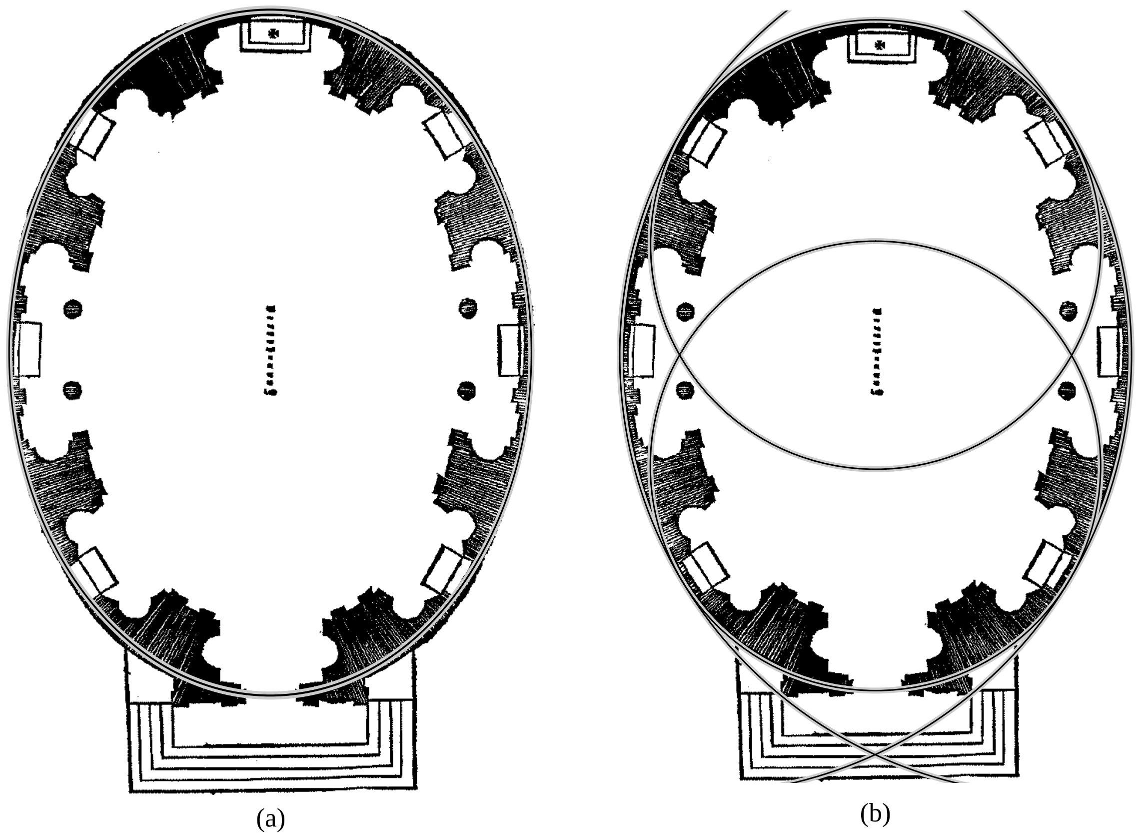

Figure 14: The best fit ellipse and oval overlaid on Serlio's church plan

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