A Survey and Comparison of Traditional Piecewise Circular Approximations to the Ellipse

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Abstract

Due to their simplicity and ease of handling quadrarcs have been used to provide a piecewise circular approximation to ellipses for many hundreds of years. Like biarcs there are many possible criteria for choosing joint positions between the arcs. This paper provides an algebraic formulation of 10 of these quadrarc constructions. For a range of ellipse eccentricities these quadrarcs are assessed quantitatively in terms of Euclidean distance error and tangent discontinuity.

keywords: ellipse, quadrarc, circular arc,

1 Introduction

The approximation of curves by piecewise analytic functions is widespread, and occurs in a diverse range of disciplines such as architecture, art, astronomy, building, and engineering. Many primitives have been used such as straight lines, conics, etc, and many techniques have been developed over the years. In this paper we will restrict our analysis to a very specific instance of this problem: the approximation of ellipses by piecewise circular arcs.

The reason that we concentrate on such a narrow field is that the ellipse is a common shape that often appears as a component of an object, especially man-made ones. Therefore ellipses have been found to be useful in the areas of CAD/CAM modelling, computer graphics, computer vision among others. In computer vision, where the images of objects are analysed, circular as well as elliptical sections are projected into the image as ellipses, emphasising their ubiquity.

Since it is easy enough to generate an exact ellipse why should it be necessary to make piecewise circular approximations? Although the ellipse has a simple enough analytic expression with linear parameters, some aspects are problematic.

- For instance, its curvature is continuously varying. Thus while circular sections can be detected in image curves relatively easily by testing for constant curvature such methods cannot be readily extended to work with ellipses.

- Some applications such as fitting ellipses to scattered data require some error estimate to be minimised. The most natural error estimate for a point is the Euclidean distance from the point along the normal to the ellipse. Unfortunately this requires solving a quartic equation, and choosing the minimum of the four solutions. In practise this is rarely carried out, and approximations to the Euclidean distance are used instead.
- The parallels of an ellipse are not ellipses themselves, but are eighth order polynomials [6]. Determining such parallels is useful in applications such as architecture which requires parallel rows of columns, etc., and in CAD where the offset curves of paths made by NC milling machines need to be determined [4].

These and other operations are much simplified when circular arcs are used as approximations; calculating normals is trivial while parallels to circular arcs are also circular.

Bolton introduced the biarc to the CAD community less than thirty years ago [3]. Since then a substantial volume of literature has flourished, relating to various alternative criteria in its construction [15, 16, 18] and its application to a variety of curves such as B-splines [14], spirals [9], and NURBS [11]. A related construct, not so well known to the CAD community, but which has been in use for centuries if not millennia, is the *quadrarc* [7] which consists of four circular arcs approximating an ellipse.[1] Some famous examples are the parallel colonnades in the Piazza del San Pietro and the parallel rows of seats in the Colosseum in Rome [7]. James Stirling applied it in the field of astronomy to aid his analysis of the elliptic orbits of planets, and it has been used for many years in masonry to help create the semi-elliptic arch, and is called the five, seven, etc centre method depending on how many arcs make up the approximation. Its importance is illustrated by the fact that in 1839 a whole book was devoted to the arch [13], and various circular approximations to the ellipse were described. A more recent example is to accurately approximate errors, enabling improved ellipse fitting [17]. Another potential use is by milling or drafting machines that are often capable of generating either straight lines and circular arcs, requiring ellipses and other curves to be approximated. As Gridgeman states [7]:

> But even with an adequate knowledge of the geometry and constructability of the ellipse, designers and artists at all times must have been tempted to use instead a quadrarc pseudoellipse because of its immeasurably superior ease of handling.

Over the years (and before the advent of computers) many ingenious manual methods were developed to construct true or approximate ellipses. Examples are the trammel, foci (pin and string), concentric (auxiliary) circle, parallel (rectangle), intersecting arc, four centre, and eight centre methods. Latterly various mechanical devices (ellipsographs) appeared in the nineteenth century [1]. The four centre method corresponds to the quadrarc, while the eight centre and other such methods approximate the ellipse by eight or more circular arcs. Looking through textbooks on technical drawing and building reveals a large number of alternative methods for constructing such approximations. Since the quadrarcs were originally intended to be constructed using a compass and ruler an important criterion for their design was their ease of manual construction. However, even though this requirement does not apply for computer design or computer vision some of the quadrarcs appear to give good approximations. Ideally we would like to find the quadrarc that produces the minimum error. Unfortunately as we will see, this cannot be easily found analytically. Therefore we will examine the various suboptimal quadrarc approximations that have been developed over the years and judge their relative merits. In order to do so we have first converted these geometric methods into their algebraic forms for more convenient analysis.

2 Quadrarc Properties

Before describing any specific quadrarcs let us start by identifying the problem. We wish to approximate an ellipse specified by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

[1] Banchoff and Giblin [2] use the term *PC ellipse* instead, and note that it is the solution to the curve of fixed length that surrounds a pair of equal radius disks and encloses the maximum area.

by a quadrarc consisting of four circular arcs with centres $(\pm h, 0)$ and $(0, \pm k)$ and radii $a - h$ and $b + k$ respectively such that they pass through the extremal points of the ellipse. Two more common additional constraints can be added concerning the joint between the arcs. To ensure tangent continuity the arcs' centres must lie on the common normal to the joint and the geometry will be as shown in figure 1. The radii are the lines $\overline{kj}$ and $\overline{hj}$ and have lengths

$$\begin{aligned} \overline{hj} &= a - h \\ \overline{hj} &= \overline{kj} - \overline{kh} = (b + k) - \sqrt{h^2 + k^2}. \end{aligned}$$

Solving these equations for h gives

$$h = \frac{k - \frac{a-b}{2}}{\frac{k}{a-b} - 1}. \tag{2}$$

Given tangent continuity the coordinates of the joint are determined by finding the intersection of the normal $\overline{kj}$ and one of the two circular arcs. For instance, taking the arc centred at $(h, 0)$ gives the following equations for the line and circle

$$\begin{aligned} y &= \frac{k}{h}x - k \\ y^2 + (x - h)^2 &= (a - h)^2 \end{aligned}$$

which are solved for (x, y) to obtain

$$\left(h \left[\frac{a - h}{\sqrt{k^2 + h^2}} + 1 \right], k \frac{a - h}{\sqrt{k^2 + h^2}} \right) \tag{3}$$

where h and k are limited to the range $h = [a - b, a]$ and $k = [\frac{(a-b)(a+b)}{2b}, \infty]$. The path of possible continuous tangent joints is given by the thick gray line in figure 2. If we substitute (2) in (3) and eliminate k we can show (after much manipulation) that the path is circular with centre $\frac{1}{2}(a - b, b - a)$ and radius $\sqrt{\frac{a^2+b^2}{2}}$.

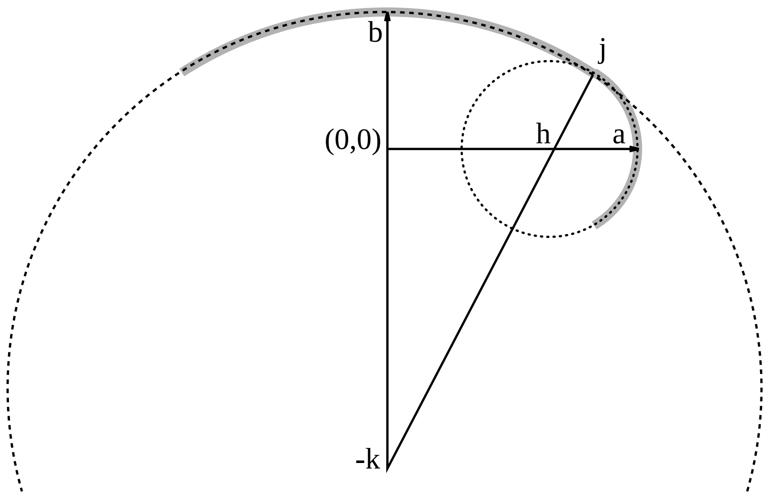


Figure 1: Arrangement of quadarc with tangent continuity

The constraint that the intersection of the arcs lie on the ellipse is considerably more complex to express. If the tangent constraint is not incorporated then the intersection of the arcs is at

$$\frac{1}{2p} \left(-a^2 h + b^2 h + 2ah^2 + 2bhk + kt, -a^2 k + b^2 k + 2bk^2 + 2ahk - ht \right) \tag{4}$$

where

$$\begin{aligned} p &= h^2 + k^2 \\ q &= a - h \\ r &= b + k \\ t &= 2hk \pm \sqrt{-p^2 - (q^2 - r^2)^2 + 2p(q^2 + r^2)}. \end{aligned}$$

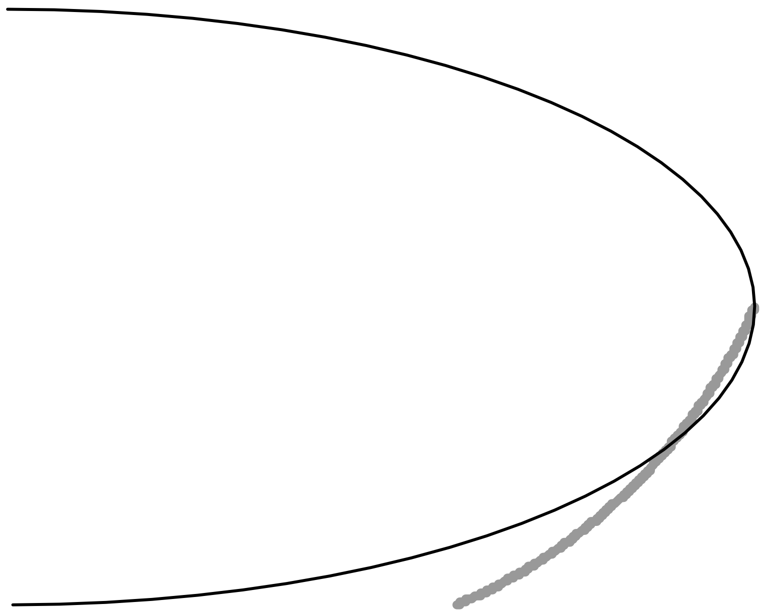


Figure 2: Locus of continuous tangent joints

Substituting (4) into (1) gives a fourth order polynomial in h and k. Although the constraint is complex, given a point on the ellipse (x, y) it is at least straightforward by rearranging (1) to determine the values of h and k such that (x, y) is the point of intersection of the arcs, namely

$$\begin{aligned} h &= \frac{a^2 - x^2 - y^2}{2(a - x)} \\ k &= \frac{b^2 - x^2 - y^2}{2(y - b)}. \end{aligned} \tag{5}$$

This will be of use later.

3 Alternative Quadrarcs

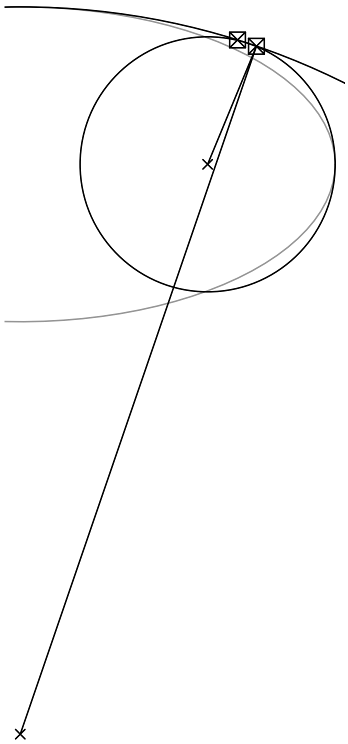
We now list several alternative quadrarcs which have been culled from a variety of books on technical drawing, building, etc. Plots of all the quadrarcs are shown in figure 3. The first example is by Knowlton [21] who describes a simple method which places the arc joint three quarters of the way along the diagonal of the bounding rectangle. The joint does not lie on the ellipse and the tangent is not continuous at the joint. The resulting values for h and k are

$$\begin{aligned} h &= \frac{7a^2 - 9b^2}{8a} \\ k &= \frac{9a^2 - 7b^2}{8b}. \end{aligned}$$

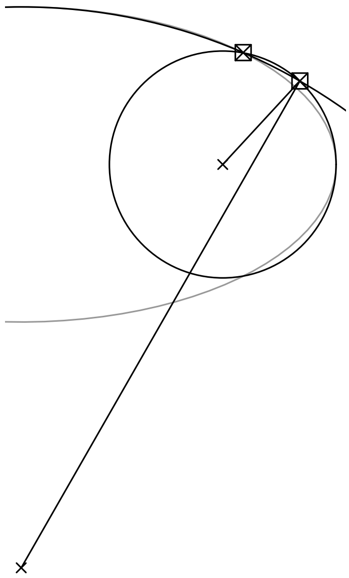
With a small modification Knowlton's method can be adapted to make the joint lie closer to the ellipse. Since the diagonals intersect the ellipse at $\frac{1}{\sqrt{2}}(a, b)$ we just shift the arc joint from 0.75 to $\frac{1}{\sqrt{2}} \approx 0.707$ along the diagonal

$$\begin{aligned} h &= \frac{a^2 - b^2}{2a(2 - \sqrt{2})} \\ k &= \frac{a^2 - b^2}{2b(2 - \sqrt{2})}. \end{aligned}$$

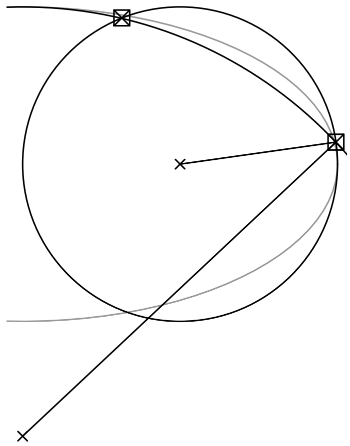
This makes one of the intersection points of the arcs lie on the ellipse. However, we are actually using the other intersection point which does not lie on the ellipse. Nevertheless, the resulting approximation is an improvement on Knowlton's basic method.



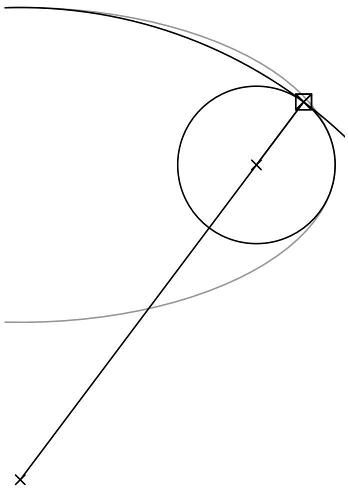
(a) Knowlton .75



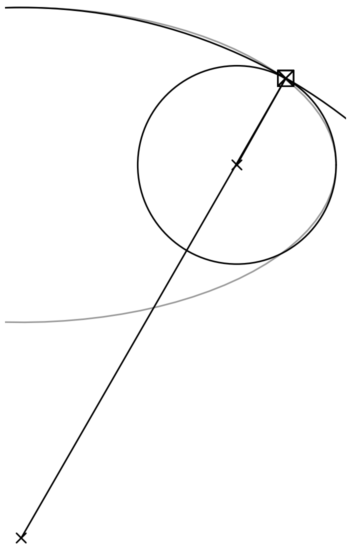
(b) Knowlton .707



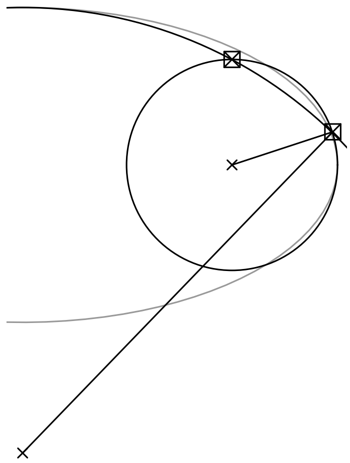
(c) Mott



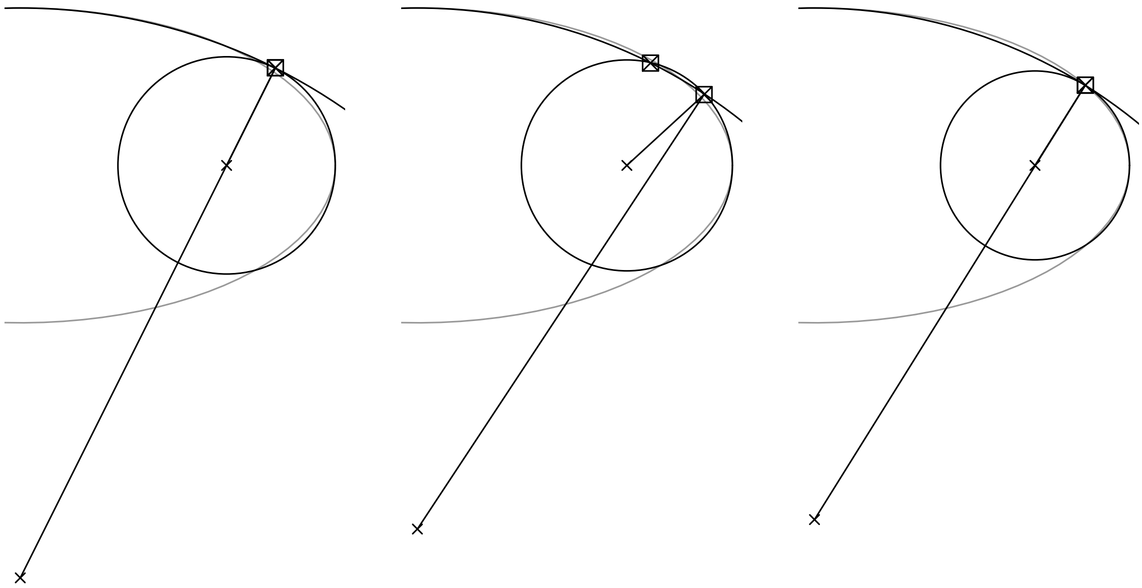
(d) French



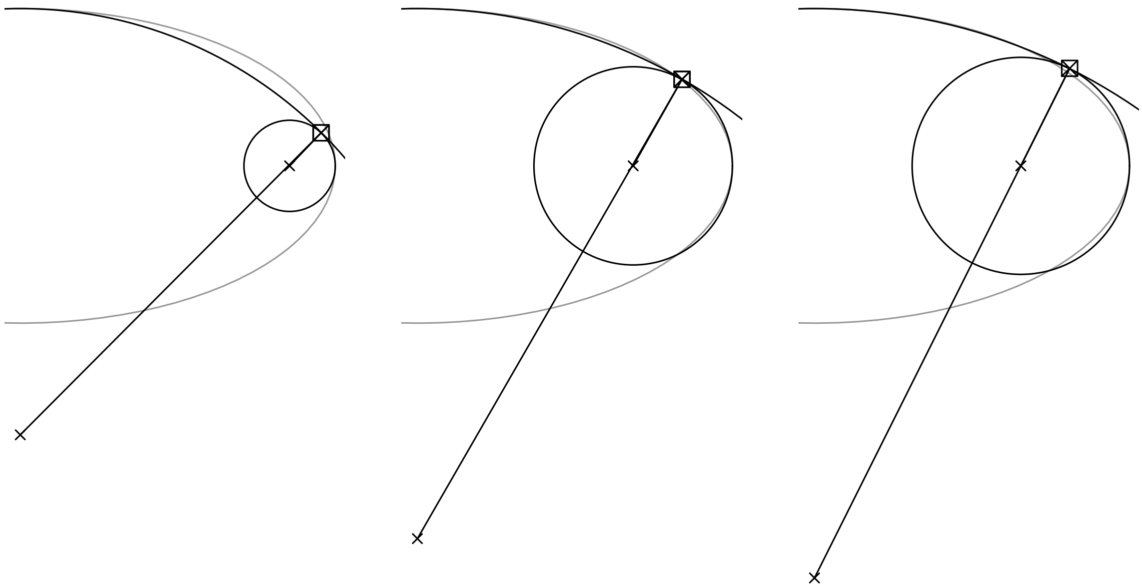
(e) Hewitt method 3



(f) Hewitt method 4



(g) Hewitt method 5 (h) Slantz (i) Stirling



(j) Gridgeman (k) optimal G^1 (l) Su & Liu biarc

Figure 3: Quadrarc circles approximating an ellipse with $\frac{a}{b} = 2$

Another simple method, given by Mott [12] (which appears there as method 7.8), fixes the smaller circles and determines the remaining two circles by constructing an equilateral triangle, giving

$$h = \frac{k}{\sqrt{3}} = \frac{a}{2}.$$

Again the joint is neither G^1 nor lying on the ellipse.

A third simple method is provided by French [5] (shown by him as figure 75) and is based on differences in the lengths of the axes

$$h = \frac{3}{4}k = \frac{3}{2}(a-b).$$

Unlike the preceding methods this one is G^1.

Hewitt's method 3 [8] is also G^1, and is based on an equilateral triangle and the intersection of various arcs and lines

$$h = \sqrt{3}k = \frac{3(a-b)}{\sqrt{3}-1}.$$

Hewitt's method 4, which is not G^1, uses the bounding rectangle and bisects and intersects various lines, resulting in

$$\begin{aligned} h &= \frac{5a^2 - 4b^2}{6a} \\ k &= \frac{4a^2 - 6b^2}{6b}. \end{aligned}$$

Hewitt's method 5 is G^1 and defines several arcs; through their intersections are constructed various lines which intersect with the axes, giving

$$\begin{aligned} h &= \frac{(a-b)(a+b+\sqrt{a^2+b^2})}{2a} \\ k &= \frac{(a-b)(a+b+\sqrt{a^2+b^2})}{2b}. \end{aligned}$$

The Slantz method [22], which is not G^1, starts with intersecting arcs followed by constructing an equilateral triangle, producing

$$h = \frac{k}{\sqrt{3}} = \frac{4}{3}(a-b).$$

Finally Gridgeman [7] provides a method using the following values for h and k

$$h = k = (1 + \frac{1}{\sqrt{2}})(a-b).$$

Although he states that they correspond to a popular construction method for G^1 quadrarcs, it does not match any of the methods I have found in the textbooks.

Regarding the two constraints on the arcs we previously mentioned we can see that the above methods satisfy either none or one of the constraints. If we incorporate both the tangent constraint (2) and also constrain the joint (3) to lie on the ellipse (1) then, apart from several degenerate solutions containing circles with zero radius, we get

$$\begin{aligned} h &= \frac{(a-b)\left(a+b+\sqrt{a^2+6ab+b^2}\right)}{a-b+\sqrt{a^2+6ab+b^2}} \\ k &= \frac{(a-b)\left(a+3b+\sqrt{a^2+6ab+b^2}\right)}{4b}. \end{aligned} \tag{6}$$

This in fact is equivalent to Stirling's approximation of an ellipse [20] which we previously used for ellipse fitting [17]. The combination of constraints should hopefully provide a visually pleasing and accurate approximation. The following section will enable us to see how close it comes to an optimal approximation.

3.1 Criteria Optimisation

We can introduce additional constraints to the ones mentioned so far. For instance, Gridgeman considers combining the tangent constraint with constraints setting the area or perimeter equal to the underlying ellipse's. A potentially more useful constraint would be to find the quadrarc with minimum error. If the joint is constrained to lie on the ellipse this simplifies the equations, but it is still difficult to perform the minimisation analytically. If the true Euclidean distance is minimised then the summed error functional involves elliptic integrals. Even if the simplest distance approximation is used instead – the algebraic distance – the expression cannot be solved analytically.

Therefore, instead we estimate the error between the ellipse and the quadrarcs by the following numerical procedure. Knowing the intersection of the two circular arcs their subtended angles can be calculated. We will denote these by θ_1 and θ_2, and their radii by $r_1 = b + k$ and $r_2 = a - h$. The pair of arcs are then divided into approximately n equal arclength sections. This is achieved by dividing the two arcs into $n_1 = \text{round}\left(\frac{nr_1\theta_1}{r_1\theta_1 + r_2\theta_2}\right)$ and $n_2 = n - n_1$ sections respectively. The arcs are stepped along from their ends, the angle increment being $\frac{\theta_1}{n_1}$ and $\frac{\theta_2}{n_2}$ respectively. At each such point on the arc we obtain an approximation e_i to the distance along the normal to the ellipse using the confocal conic method [17]. This approximate distance is then used to generate the various error norms; i.e. $L_1 = \sum_i e_i$, $L_2 = \sum_i e_i^2$, and $L_\infty = \max e_i$. In our experiments we set $n = 1000$, but little difference was found when n was increased or decreased by a factor of ten.

To determine the optimal tangent continuous quadarc for a particular pair of values of a and b we perform a 1D search over the range of h (and the associated k) values satisfying (2) to find the minimum error. The result of $\frac{a}{b} = 2$ plotted in figure 3 shows that this produces an extremely accurate approximation. Furthermore, the error plots in figure 7 confirm that the optimal G^1 does not correspond to any of the methods from the literature that we have considered here.

Similarly, a 1D search is sufficient to numerically estimate the optimal joint constrained to lie on the ellipse but not constrained to be G^1. We step along the ellipse using the parametric form of (1)

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}$$

and calculate the appropriate values of h and k at each point using (5). We always choose the intersection point furthest from the ellipse centre. Since it is sometimes the other intersection point that lies on the ellipse, then the constraint can fail. This occurs when the circle at the pointed end becomes bigger than the tangential circle calculated using Stirling's method, see (6), i.e.

$$k > \frac{(a-b)\left(a + 3b + \sqrt{a^2 + 6ab + b^2}\right)}{4b}.$$

In fact, on performing the minimisation we found the optimum error occurs when the two arcs have tangent continuity. In other words, Stirling's method provides the minimum error under the constraint that the joint lies on the ellipse.

In summary, the quadrarc construction methods can be classified according to the two constraints considered in this paper. There are four possible combinations of the constraints being met or not, as shown in table 1. The classification of the various quadrarc methods using this scheme is shown in table 2. It is interesting to note that there are no methods of type III.

Type	Joint on Ellipse	Tangent Continuity of Arcs
I	$\times$	$\times$
II	$\times$	$\surd$
III	$\surd$	$\times$
IV	$\surd$	$\surd$

Table 1: Classification of quadrarc constraints

Method	Type
Knowlton .75	I
Knowlton .707	I
Mott	I
French	II
Hewitt method 3	II
Hewitt method 4	I
Hewitt method 5	II
Slantz	I
Stirling	IV
Gridgeman	II
optimal G^1	II
Su & Liu biarc	II

Table 2: Classification of quadrarc methods

3.2 Caveats

For those methods without tangent continuity (types I and III) there is the question of which intersection point of the two arcs to use as the joint. We choose the point closer to the pointed end of the ellipse since it provides a more pleasing curve, avoiding the concavities resulting from the other intersection point. However, this does not necessarily provide a more accurate approximation. For instance, figure 4 shows that using Hewitt's method 4 the error curves cross over such that for moderately low eccentricity the more central intersection point produces a lower error.

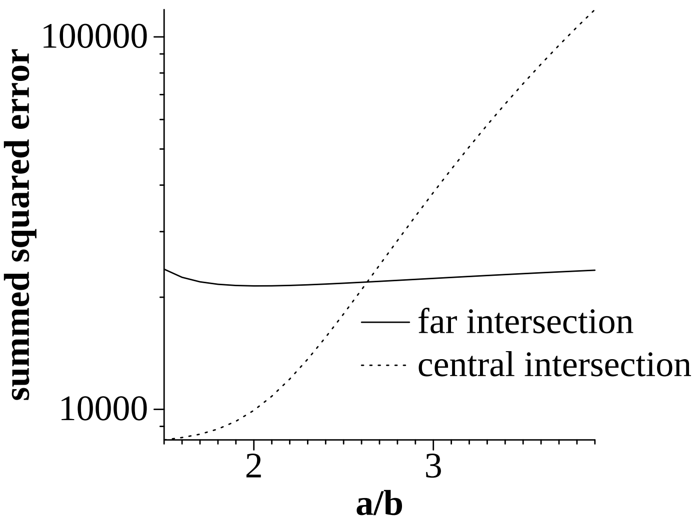


Figure 4: Error crossover for different arc intersection points

Although not mentioned in any of the textbooks cited an additional problem for those methods without tangent continuity is that, depending on the eccentricity of the ellipse, it is possible for the intersection point to lie on the wrong side of the major axis, thereby invalidating the approximation. An example is given in figure 5. To find the breakdown point we solve for the general intersection point setting $y = 0$ in (4). The various ranges of failure are given in table 3.

3.3 Biarcs

To conclude the catalogue of quadrarcs the biarc is included as a more contemporary method of construction. The biarc is constrained to pass through two points with specified tangents at these points. In addition the arcs joint is G^1. These five conditions leave one degree of freedom which can be eliminated using many different criteria [3, 19]. As an example we use the method given

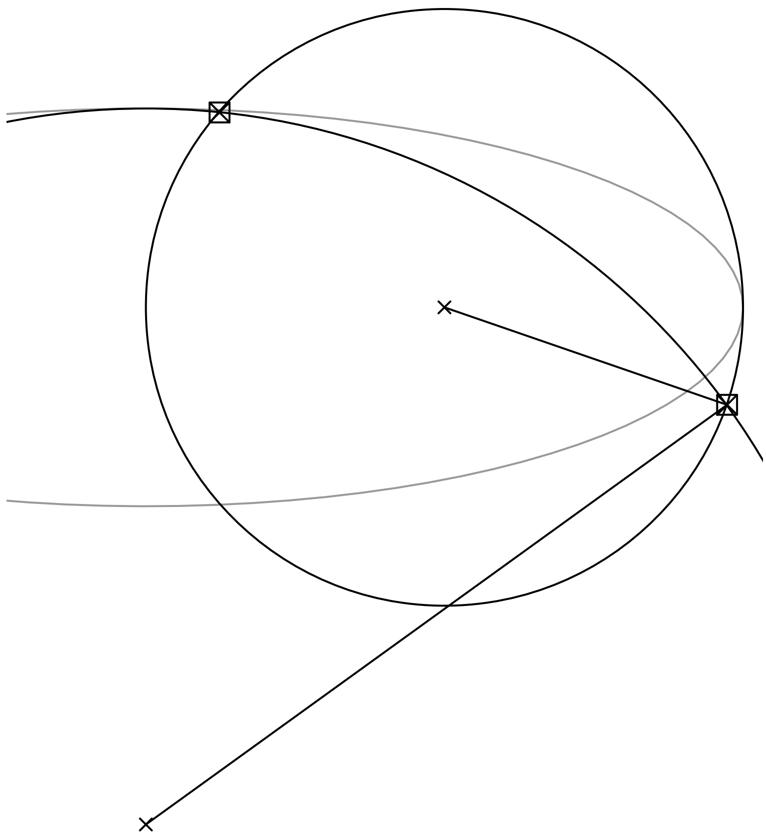


Figure 5: Breakdown of Mott's method when $\frac{a}{b} = 3$

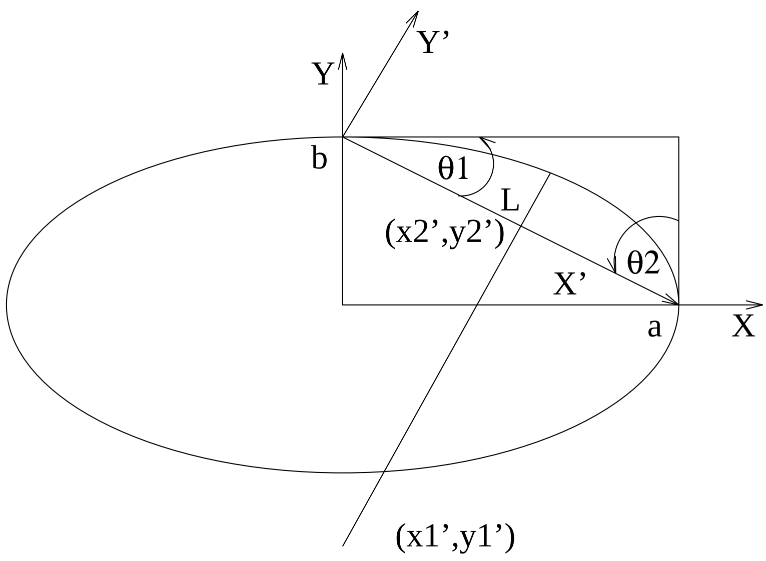


Figure 6: Construction of Su & Liu biarc

Method	breakdown range
Mott	$\frac{a}{b} > \frac{\sqrt{7}+\sqrt{3}}{2} \approx 2.1889$
French	$\frac{a}{b} > 3$
Hewitt 4	$\frac{a}{b} < \sqrt{2} \approx 1.4142$
Slatz	$\frac{a}{b} > \frac{8}{\sqrt{3}} - 1 \approx 3.6188$
Gridgeman	$\frac{a}{b} > 1 + \sqrt{2} \approx 2.4142$

Table 3: Breakdown points of quadrarc methods

by Su and Liu which keeps the ratio of the arc radii as close as possible to one. The construction follows figure 6. For an ellipse we can readily calculate:

$$\begin{aligned}\theta_1 &= tan^{-1}\frac{b}{a}\\ \theta_2 &= tan^{-1}\frac{a}{b}\\ L &= \sqrt{a^2+b^2}\end{aligned}$$

which enables the radii of the arcs to be determined as

$$\begin{aligned}R_1 &= \frac{L\sin\frac{\theta_2}{2}}{2\sin\frac{\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}}\\ R_2 &= \frac{L\sin\frac{\theta_1}{2}}{2\sin\frac{\theta_2}{2}\sin\frac{\theta_1-\theta_2}{2}}.\end{aligned}$$

In the coordinate frame $X'Y'$ the coordinates of the arc centres are

$$\begin{aligned}(x_1, y_1) &= (-R_1\sin\theta_1, R_1\cos\theta_1)\\ (x_2, y_2) &= (-R_1\sin\theta_1, R_2\cos\theta_2).\end{aligned}$$

These are transformed back to the XY axes after a rotation of $-\theta_1$ and translation of $(0, b)$, giving (x'_1, y'_1) and (x'_2, y'_2). Finally, converting to our previous specification of quadrarcs,

$$\begin{aligned}h &= x'_2\\ k &= y'_1.\end{aligned}$$

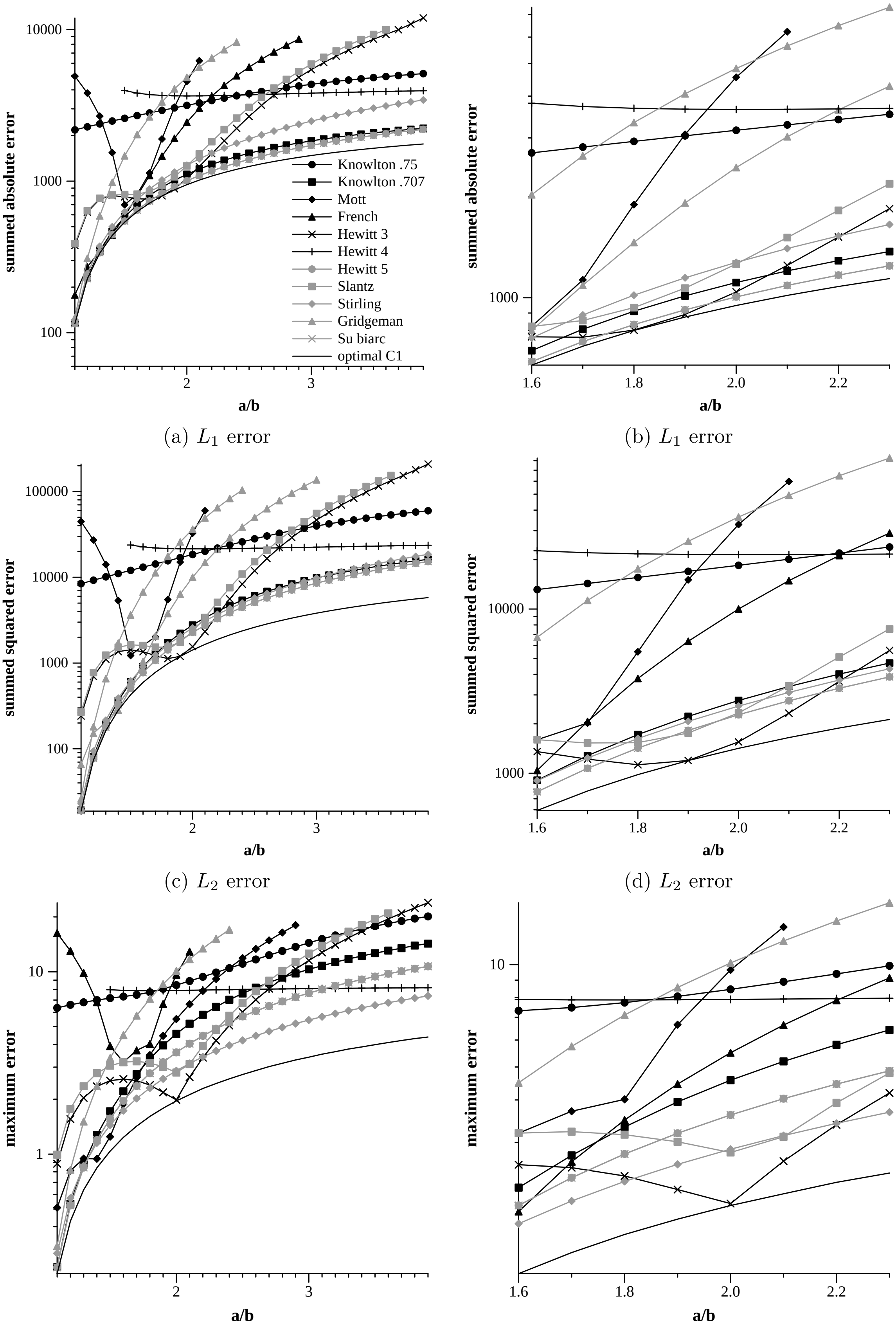
4 Performance Analysis

Figure 7 shows the estimated errors incurred by the various quadrarcs using the L_1, L_2, and L_∞ norms. The first point to note is that none of the traditional methods are optimal, although some get close. We expected that Stirling's quadrarc should fare well given that it is G^1 and the joint lies on the ellipse. It does, but is still outperformed by several others in terms of L_1 and L_2 error. Even though it satisfied neither of these constraints, Knowlton .707 consistently received a lower L_1 error. Hewitt 5 performed better on both L_1 and L_2 error and gave identical results to Su & Liu's biarc. On occasion did Hewitt 3 and Slantz also performed well but degraded badly for highly eccentric ellipses. Some of the other methods such as Mott, French, and Gridgeman degraded even earlier, at only moderate eccentricity. Knowlton .75 and Hewitt 4 never perform well.

The results for the L_∞ error are more spread out. Hewitt 3 does very well here (and on the other error norms too) at $\frac{a}{b} \approx 2$. Apart from that, Stirling usually fares best, with Hewitt 5 and then Knowlton .707 as the next best contenders.

We note that that Su & Liu's biarc gave identical results to Hewitt's method 5, although it is difficult

To measure the tangent discontinuity incurred by some of the quadrarcs the difference in the angle of the normals at the joint is shown in figure 8. We can see that despite the low approximation error shown by Knowlton .707 the discontinuity becomes considerable as the ellipse eccentricity increases.



(a) L_1 error (b) L_1 error

(c) L_2 error (d) L_2 error

(e) L_∞ error (d) L_∞ error

Figure 7: Errors for quadrarcs according to L_1, L_2, and L_∞ norms

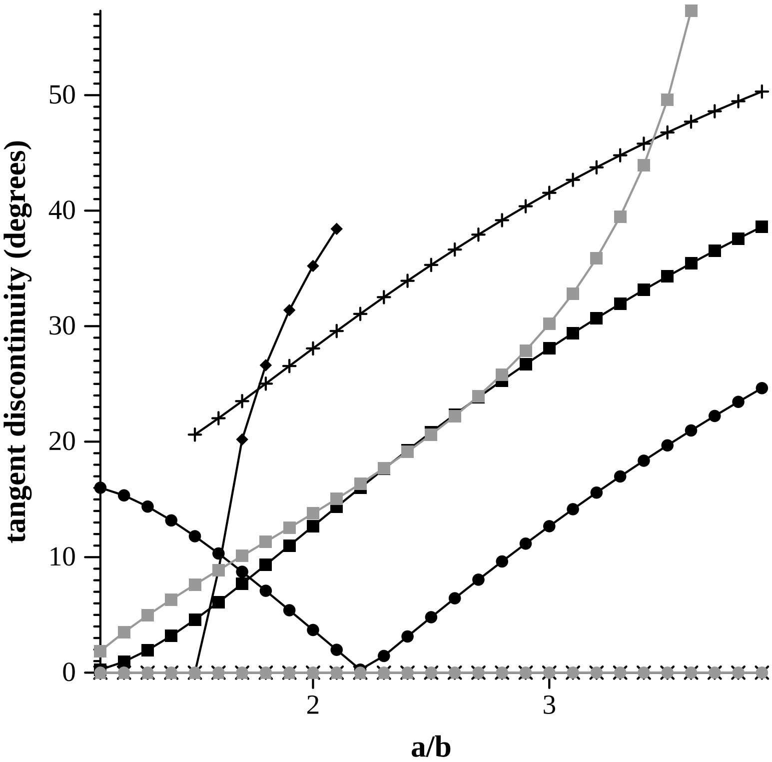


Figure 8: Tangent discontinuity

5 Using Additional Arcs

Having covered a considerable range of quadrarc approximations a logical progression would be to improve the ellipse approximation by using more than four circular arcs. Again the technical drawing and building literature provides some examples, although rather fewer in number than quadrarc construction methods. An example of a G^1 eight arc approximation is given by McKay [10]. Four arcs corresponding to the quadrarc elements are used as before, and are calculated as

$$\begin{aligned} h &= \sqrt{3}(a-b) \\ k &= 2(a-b), \end{aligned}$$

and four additional arcs with centres $(\pm x_1, \pm y_1)$ and radius r_1 are inserted

$$\begin{aligned} x_1 &= \frac{\sqrt{3}}{2}(a-b) \\ y_1 &= -\frac{a-b}{2} \\ r_1 &= 2a-b-\sqrt{3}(a-b). \end{aligned}$$

Again, care must be taken as the method breaks down at $\frac{a}{b} > \frac{\sqrt{3}}{\sqrt{3}-1} \approx 2.366$ when the circle furthest from the ellipse centre shrinks to zero radius.

An alternative eight arc approximation (which is not G^1) is given by Nicholson [13] (in article 6). It breaks down for low eccentricity ellipses as the inner two circles do not intersect, but formulating the exact conditions is rather complex [13]

$$\begin{aligned} h &= \frac{a^2-b^2}{a} \\ k &= \frac{a^2-b^2}{b} \\ x_1 &= \frac{(a^2-b^2)(2a-b)}{a(3a-b)} \end{aligned}$$

$$y_1 = -\frac{a^2 - b^2}{2(3a - b)}$$

$$r_1 = \frac{b^2}{a} + \frac{\sqrt{5}(a^2 - b^2)}{2(3a - b)}.$$

Nicholson extends the method to find a twelve arc circle in article 7, and describes how it can made to generate any number of subarcs. However, we have found difficulties with it, and find that the inner circles rarely, if ever, intersect.

Surprisingly, when we plot their errors (see figure 9) we see that the eight centred approximations do not seem to offer any advantages over quadrarcs. McKay's method is clearly inferior to many quadrarcs while Nicholson's performs at a comparable level to the best quadrarcs. An example of the component circles are shown in figure 10.

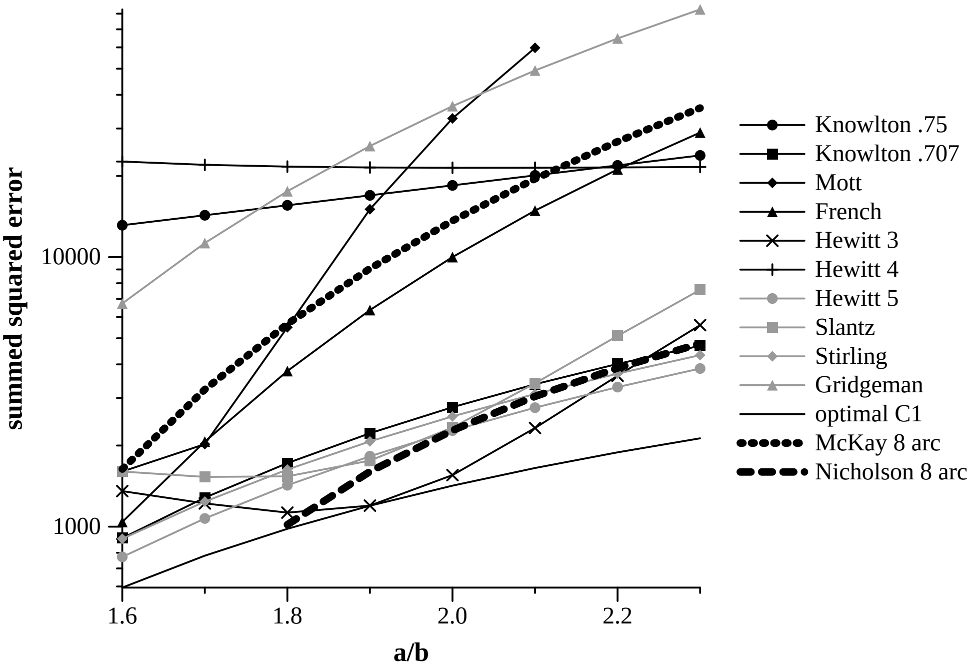


Figure 9: Errors for quadarcs and eight centre approximations (L_2 norm)

6 Conclusion

A wide selection of quadrarcs have been analysed to assess their suitability for approximating ellipses. Stirling's method has the advantage of possessing tangent continuity and has a low maximum deviation at all eccentricities. However, the simpler Knowlton .707 and Hewitt 5 methods are also good candidates. In particular the Hewitt 5 method has lower L_1 and L_2 errors than Stirling's method, and also has continuous tangents. Unlike some of the other methods considered, none of the above break down for extreme eccentricities. Surprisingly, these quadrarcs perform sufficiently well that the eight arc approximations that we tested provide no advantages.

Experimentally it was found that Su & Liu's biarc was identical to Hewitt's method 5 although their construction methods differ. In the future it would be interesting to analyse alternative biarcs constructions to compare them with the quadrarcs described in this paper.

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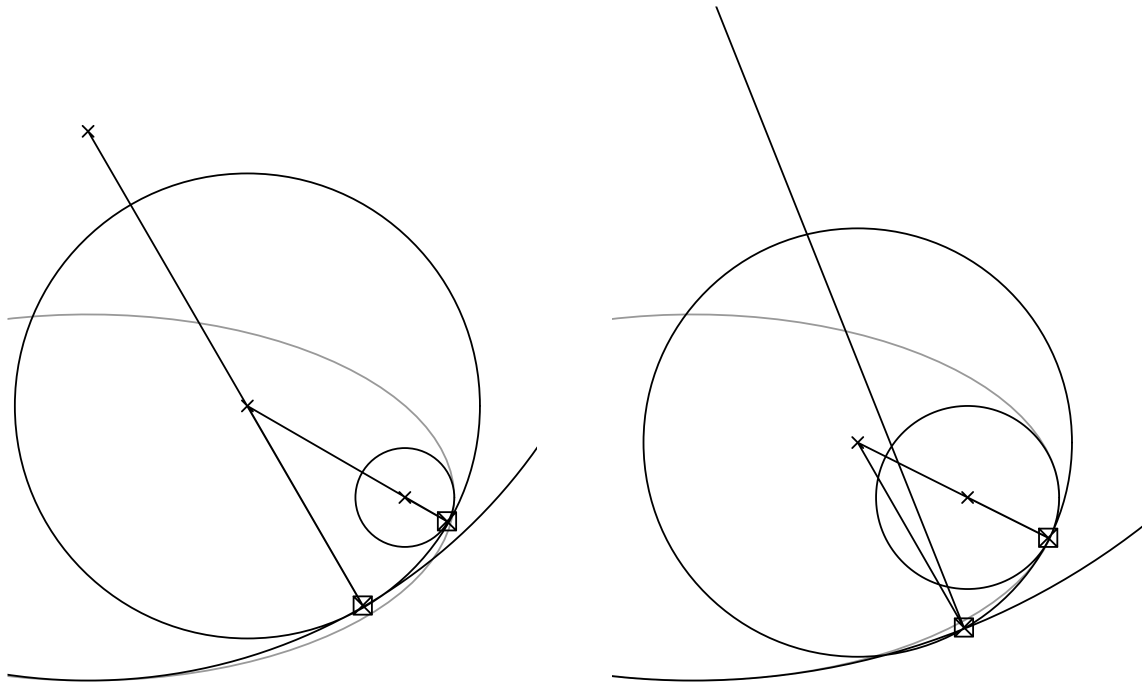

(a) McKay

(b) Nicholson

Figure 10: Eight centre approximations for an ellipse with $\frac{a}{b} = 2$

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