Shape Orientability

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Abstract. In this paper we consider some questions related to the orientation of shapes. We introduce as a new shape feature *shape orientability*, i.e. the degree to which a shape has distinct (but not necessarily unique) orientation. A new method is described for measuring shape orientability, and has several desirable properties. In particular, unlike the standard moment based measure of elongation, it is able to differentiate between the varying levels of orientability of n-fold rotationally symmetric shapes.

1 Introduction

This paper deals with some of the problems, and proposes solutions, related to shape *orientability* – i.e. the degree to which shape has distinct (but not necessarily unique) orientation. The computation of a shape's orientation is a common task in the area of computer vision and image processing, being used for example to define a local frame of reference, and helpful for recognition and registration, robot manipulation, etc.

There are situations (see Fig. 1) when the orientation of the shapes seems to be easily and naturally determined. On the other hand, a planar disc could be understood as a shape without orientation.

Most situations are somewhere in between. For very non-regular shapes it could be difficult to say what the orientation should be. Rotationally symmetric polygons could also have poorly defined orientation – see Fig 2 (d). Moreover, even for regular polygons (see Fig. 2 (a) and (b)) is debatable whether they are orientable or not. For instance, is a square an orientable shape? The same question arises for any regular n-gon, but also for shapes having several axes of symmetry, and n-fold ($n > 2$) rotational symmetric shapes. If the answer is "yes, those shapes are somehow orientable", how should the shapes from Fig. 2 be ranked with respect to their orientability? This question is of interest and applicable in the area of shape analysis and shape classification.

The most standard method for computing shape orientability (derived in section 2 and specified in eqn (5)) is based on computing the axis of the least second moment. It is naturally defined and easy to compute. However, it does not specify what the shape orientation should be in those examples (see section 2).

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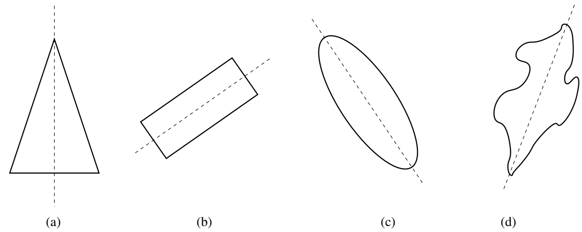

Fig. 1. Reasonable orientations of the shapes coincide with the dashed lines

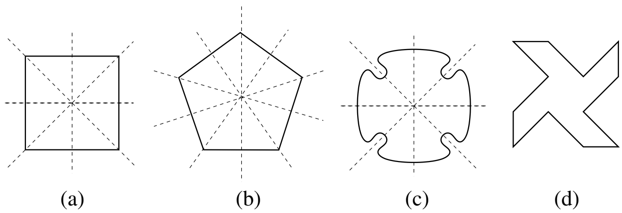

Fig. 2. Reasonable orientations of the shapes coincide with the dashed lines

The problem becomes more complex taking into account that in computer vision and image processing tasks real shapes are replaced with their digitizations. Some specific problems arise when working with digital shapes. Let us mention just two of them:

– Due to the digitization process some "non-orientable" objects may have digitizations whose orientation can be easily computed if (5) is applied.
– On the other hand, it is also possible that some orientable objects have digitalizations which are not orientable.

The impact of digitization effects on changing the computed shape orientation is illustrated by the example of a digitized disc and a digitized square. Even though real discs and squares are not "orientable" shapes (if the standard method is applied – see Lemma 1) it could happen that after digitization, the obtained discrete point sets have an orientation computable in the standard manner. We demonstrate that the computed orientation could depend strongly on:

(a) shape position with respect to the digitization grid;
(b) applied picture resolution.

The effect of item (a) is illustrated by Fig. 3. The same disc is translated into 3 different positions and then digitized. The orientation of the digital disc is not well-defined (in the sense of (5)) for the position displayed in Fig. 3 (a) while the digital discs displayed at Fig. 3 (b) and (c) have the measured orientation $\varphi = \pi/2$ – if (5) is applied. If the applied picture resolution is higher (or equivalently, a bigger disc is digitized) then the impact of the disc position to the computed orientation is higher, as well. As an illustration: we have digitized 16 real discs having the radius equal to 10, whose center positions have been chosen randomly.

For each choice of center position the computed orientations of the obtained digital disc (applying formula (5)) (in the range $[-\pi/2, \pi/2]$) are

0.05	0.03	-0.06	-0.59	0.75	-0.01	-0.23	-0.72
0.13	0.00	0.22	-0.57	-0.06	0.29	-0.61	0.63

and show that the computed orientation strongly depends on the disc position with respect to the digitization grid.

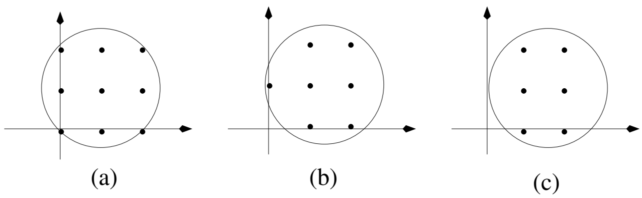

Fig. 3. Three of the 6 non-isometric digitizations of a disc having the radius $\sqrt{2}$ on a binary picture with resolution 1 (i.e., one pixel per measure unit)

Similar problems to the above ones can be caused by noise effects, as well. For instance, consider a square aligned with the coordinate axes. As mentioned, the standard method does not give any answer what the orientation of such a square should be. Adding a single protruding pixel to the boundary can cause the computed orientation to lie anywhere in the range $[-\pi/2, \pi/2]$ depending on its location. As an example, for a 10×10 grid of pixels adding one pixel to the horizontal or vertical edge gives the following computed orientations

0.88	1.00	1.14	1.30	1.48	-1.48	-1.30	-1.14	-1.00	-0.88
-0.69	-0.57	-0.43	-0.27	-0.09	0.09	0.27	0.43	0.57	0.69.

In order to avoid the previously mentioned problems it is not enough to determine if the orientation can be computed or not. It would be useful to see how stable the solution is. For this purpose we will define the *shape orientability* as a shape descriptor. The main purpose of it is to suggest an answer to the question: *Is the computed orientation just a consequence of digitization or noise effects or is it an inherent property of the considered shape?* The orientability can also be used as a shape descriptor in shape classification tasks.

In this paper we will define an orientability measure, which is a number from [0, 1). The defined orientability measure says that a circle has the lowest measured orientability equal to 0. Also, there is no a shape with the measured orientability equal to 1, but shapes having the measured orientability arbitrarily close to 1 can be constructed easily. For example, a rectangle with the edge lengths 1 and a has orientability tending to 1 if $a \rightarrow \infty$. This new measure will be described in Section 3. Some experimental results are shown in Section 4, while Section 5 contains concluding remarks.

2 Standard Method

In this section we give a short overview of the method which is mostly used in practice and give a lemma that shows that this method can not be understood as efficient when applied to shapes that have several axes of symmetry.

The standard approach defines the orientation by the so called axis of the least second moment ([3, 4]). That is the line which minimises the integral of the squares of distances of the points (belonging to the shape) to the line. The integral is

$$I(S,\varphi,\rho) = \iint_S r^2(x,y,\varphi,\rho)dxdy \tag{1}$$

where $r(x,y,\varphi,\rho)$ is the perpendicular distance from the point (x,y) to the line given in the form

$$x \cdot \cos\varphi - y \cdot \sin\varphi = \rho.$$

It can be shown that the line that minimizes $I(S,\rho,\varphi)$ passes through the centroid $(x_c(S),y_c(S))$ of the shape S where $(x_c(S),y_c(S)) = \left(\frac{\iint_S xdxdy}{\iint_S dxdy}, \frac{\iint_s ydxdy}{\iint_S dxdy}\right)$. In other words, without loss of generality, we can assume that the origin is placed into the centroid, but also, that the required line minimizing $I(S,\rho,\varphi)$, passes through the origin – i.e., we can set $\rho = 0$. In this way, the shape orientation problem can be reformulated to the problem of determining φ for which the function $F(\varphi,S)$ defined as

$$F(\varphi,S) = I(S,\varphi,\rho=0) = \iint_S (x\cdot\sin\varphi - y\cdot\cos\varphi)^2 dxdy^1$$

reaches the minimum. Once again, we assume that the origin coincides with the center of gravity of S.

Further, if the central geometric moments $\overline{m}_{p,q}(S)$ are defined as usual by:

$$\overline{m}_{p,q}(S) = \iint_S (x - x_c(S))^p \cdot (y - y_c(S))^q \; dx\; dy,$$

and since $(x_c(S),y_c(S)) = (0,0)$ is assumed, we have

$$\begin{aligned} F(\varphi,S) \;=\; & (\sin\varphi)^2 \cdot \overline{m}_{2,0}(S) - \sin(2\cdot\varphi)\cdot\overline{m}_{1,1,}(S) \\ & +(\cos\varphi)^2 \cdot \overline{m}_{0,2}(S). \end{aligned} \tag{2}$$

The minimum of the function $F(\varphi,S)$ can be computed easily. Setting the first derivative $F'(x,S)$ to zero, we have

$$F'(\varphi,S) = \sin(2\varphi)\cdot(\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)) - 2\cdot\cos(2\varphi)\cdot\overline{m}_{1,1}(S) = 0.$$

[1] The squared distance of a point (x,y) to the line $X\cdot\cos\varphi - Y\cdot\sin\varphi = 0$ is $(x\sin\varphi - y\cos\varphi)^2$.

That easily gives that the required angle φ, but also the angle $\varphi + \pi/2$, satisfies the equation

$$\frac{\sin(2\varphi)}{\cos(2\varphi)} = \frac{2 \cdot \overline{m}_{1,1}(S)}{\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)}. \tag{3}$$

Consequently, the maximum and minimum of $F(S, \varphi)$ are as follows

$$\max\{F(S,\varphi) \mid \varphi \in [0, 2\cdot\pi]\} = \frac{1}{2} \cdot (\overline{m}_{2,0}(S) + \overline{m}_{0,2}(S))$$

$$+\frac{1}{2} \cdot \sqrt{4 \cdot \overline{m}_{1,1}(S) + (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2},$$

and

$$\min\{F(S,\varphi) \mid \varphi \in [0, 2\cdot\pi]\} = \frac{1}{2} \cdot (\overline{m}_{2,0}(S) + \overline{m}_{0,2}(S))$$

$$-\frac{1}{2} \cdot \sqrt{4 \cdot \overline{m}_{1,1}(S) + (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2}.$$

The ratio between $\max\limits_{\varphi\in[0,\pi)} F(S,\varphi)$ and $\min\limits_{\varphi\in[0,\pi)} F(S,\varphi)$

$$\mathcal{E}(S) = \frac{\max\{F(S,\varphi) \mid \varphi \in [0, 2\cdot\pi]\}}{\min\{F(S,\varphi) \mid \varphi \in [0, 2\cdot\pi]\}} \tag{4}$$

is well known as the *elongation* of the shape S.

Let us mention that, when working with digital objects which are actually digitalizations of real shapes, then central geometric moments $\overline{m}_{p,q}(S)$ are replaced with their discrete analogue, i.e., with so called, *central discrete moments.* Since the digitization on the integer grid $\mathbf{Z}^2$ of a real shape S consists of all pixels whose centers are inside S it is natural to approximate $\overline{m}_{p,q}(S)$ by the central discrete moment $\overline{\mu}_{p,q}(S)$ which is defined as

$$\overline{\mu}_{p,q}(S) = \sum_{(i,j)\in S\cap\mathbf{Z}^2} (i - x_{cd}(S))^p \cdot (j - y_{cd}(S))^q\,,$$

where $(x_{cd}(S), y_{cd}(S)) = \left(\frac{\sum\limits_{(x,y)\in S\cap\mathbf{Z}^2} x}{\sum\limits_{(x,y)\in S\cap\mathbf{Z}^2} 1}, \frac{\sum\limits_{(x,y)\in S\cap\mathbf{Z}^2} y}{\sum\limits_{(x,y)\in S\cap\mathbf{Z}^2} 1} \right)$ is the centroid of discrete shape $S \cap \mathbf{Z}^2$.

Some answers about the efficiency of the approximation $\overline{m}_{p,q}(S) \approx \overline{\mu}_{p,q}(S)$ can be found in [5].

If the geometric moments in (3) are replaced with the corresponding discrete moments we have the equation

$$\frac{\sin(2\varphi)}{\cos(2\varphi)} = \frac{2 \cdot \overline{\mu}_{1,1}(S)}{\overline{\mu}_{2,0}(S) - \overline{\mu}_{0,2}(S)} \tag{5}$$

which describes the angle φ which is used as an approximate orientation of the shape S, i.e., the angle which is used to describe the orientation of discrete shape

$S \cap \mathbf{Z}^2$. It is worth noting that equation (5) can be derived easily if the orientation of the discrete set (a finite number point set) $S \cap \mathbf{Z}^2$ is defined by the line (passing the origin) which minimizes the total sum $\sum_{(i,j)\in S\cap\mathbf{Z}^2}(i \cdot \sin\varphi - j \cdot \cos\varphi)^2)$ of squares of distances of points from $S \cap \mathbf{Z}^2$ to this line.

In other words, the equality (5) can be derived as a consequence when trying to solve the following optimization problem

$$\min \left\{ \sum_{(i,j)\in S\cap\mathbf{Z}^2} (i \cdot \sin\varphi - j \cdot \cos\varphi)^2 \mid \varphi \in [0, \pi] \right\} \tag{6}$$

assuming that the centroid $\left(x_{cd}(S \cap \mathbf{Z}^2),\ y_{cd}(S \cap \mathbf{Z}^2)\right)$ coincides with the origin.

So, the standard method is very simple (in both “real” and “discrete” versions) and it comes from a natural definition of the shape orientation. However, it is not always effective. The next lemma shows that the method does not always give a clear answer what the shape orientation should be – for more details see [9].

Lemma 1. *If a given shape S has more than two axes of symmetry then $F(\varphi, S)$ is a constant function.*

Proof. From (3) it is obvious that the function $F(\varphi, S)$ could have exactly one maximum and one minimum on the interval $[0, \pi)$, or it must be a constant function. Trivially $F(0, S) = F(\pi, S)$. So, if S has more than two axes of symmetry it must be constant since $F'(\varphi, S)$ does not have more than two zeros on the interval $[0, \pi)$. ▮

Remark. A direct consequence of Lemma 1 is that

– $F(S, \varphi) = \frac{1}{2} \cdot (\overline{m}_{2,0}(S) + \overline{m}_{0,2}(S))$ for all $\varphi \in [0, \pi)$;
– $\mathcal{E}(S) = 1$

holds for all shapes that have more than two axes of symmetry. In other words, the standard method does not specify the orientation of shapes from Fig. 2, or more generally, what the orientation is for shapes having more than two axes of symmetry. Also, for all such shapes the measured elongation is 1 – i.e., the same as the measured elongation for a circle, what is not a desirable property.

3 Measuring Shape Orientability

In this section we consider what quantity can be used to describe shape orientability – to be used as an inherent shape property.

Intuitively, it can be assumed that shapes with high measured elongation are more orientable than shapes with lower measured elongation. Thus, the elongation $\mathcal{E}(S)$ (see (4)) can be used to estimate shape orientability. Since $\mathcal{E}(S) \in [1, \infty)$, in order to have the measured orientability between 0 and 1, we can measure the orientability as:

$$1 - \frac{1}{\mathcal{E}(S)}. \tag{7}$$

Several other measures can be derived from the function $F(S, \varphi)$, as well. For example, a larger ratio between the areas of the regions bounded by:

- the coordinate axes, line $y = \min_{\varphi \in [0,\pi)} F(S, \varphi)$, and line $x = \pi$, and
- the coordinate axes, line $y = F(S, \varphi)$, and line $x = \pi$,

should indicate a lower shape orientability. This leads to the following:

Definition 1. *For a given shape S its orientability $\mathcal{D}_F(S)$ can be measured as*

$$\mathcal{D}_F(S) = 1 - \frac{\pi \cdot \min\{F(S, \varphi) \mid \varphi \in [0, \pi)\}}{\int_0^\pi F(S, \varphi) \cdot d\varphi}$$

$$= \frac{\sqrt{4 \cdot (\overline{m}_{1,1}(S))^2 \ + \ (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2}}{\overline{m}_{2,0}(S) + \overline{m}_{0,2}(S)}.$$

Obviously, $\mathcal{D}_F(S)$ is easily computable and well-motivated. However, it is clear that all shape orientability measures based on $F(S, \varphi)$ are limited by the result of Lemma 1, i.e., $\mathcal{D}_F(S) = 1 - 1/\mathcal{E}(S) = 0$ for all shapes S having more than two axes of symmetry. In some situations (applications) a new measure for shape orientability is required that does not have that disadvantage.

Now, we define such a measure. When dealing with shapes that have several axes of symmetry, such shapes do not necessarily have identical measured orientability, as would result when using $1 - 1/\mathcal{E}(S)$ and $\mathcal{D}_F(S)$, for example.

Definition 2. *For a given shape S let $R(\alpha)$ be the minimal rectangle whose edges make an angle α with the coordinate axes and which includes S (see Fig. 4). Let the following hold:*

$$\mathcal{A}_{min}(S) \ = \ \min_{\alpha \in [0,\pi)} \{ \ Area\_of\_R(\alpha) \ \},$$

$$\mathcal{A}_{max}(S) \ = \ \max_{\alpha \in [0,\pi)} \{ \ Area\_of\_R(\alpha) \ \}.$$

Then, we define the orientability measure $\mathcal{D}(S)$ of the shape S as:

$$\mathcal{D}(S) \ = \ 1 - \frac{\mathcal{A}_{min}(S)}{\mathcal{A}_{max}(S)}.$$

The next theorem describes some desirable properties of $\mathcal{D}(S)$. Because of simplicity, the proof is omitted.

Theorem 1. *The new defined measure for the shape orientability has the following properties:*

- $\mathcal{D}(S) \in [0, 1)$ *for any shape S;*
- *A circle has the measured orientability equal to* 0*;*
- *The measured orientability is invariant w.r.t. similarity transformations.*

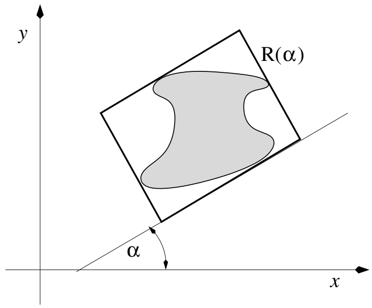

Fig. 4. The rectangle $R(\alpha)$ is the minimum area rectangle which includes the given shape (dashed area) and whose edges make an angle α with the coordinate axes

The new orientability measure introduced by Definition 2 is very convenient for numerical computation with arbitrary precision. The exact computation of $\mathcal{D}(S)$ when the measured shape S is a polygon will be described in detail in a forthcoming publication by the authors. Note that the problem of computation of $\mathcal{A}_{min}(S)$ is well studied in literature. It has been shown [2] that for a given polygon S a rectangle which has the minimal possible area and which includes the polygon S must have an edge parallel to an edge of the convex hull of S. An efficient, linear time, algorithm for such a computation (if S is a simple polygon) has been described in [8], using the technique of orthogonal calipers.

The main objection to $\mathcal{D}(S)$ is that shapes having the same convex hull have the same measured orientability. The following slight modification of Definition 2 ensures that a given non-convex shape does not have the measured orientability equal to the measured orientability of its convex hull.

Definition 3. *For a given shape S let $R(\alpha)$, $\mathcal{A}_{min}(S)$, and $\mathcal{A}_{max}(S)$ be defined as in Definition 2. Then, for any real number $\alpha \in [0,1)$ we define the orientability measure $\mathcal{D}_\alpha(S)$ of the shape S as:*

$$\mathcal{D}_\alpha(S) \;=\; 1 - \frac{\mathcal{A}_{min}(S) - \alpha \cdot Area\_of\_S}{\mathcal{A}_{max}(S) - \alpha \cdot Area\_of\_S}.$$

Note that the orientability measure $\mathcal{D}_\alpha$ also has the desirable properties listed in Theorem 1.

4 Some Examples

We now give some examples of orientability calculated using the new measure. The first example (see Fig. 5) shows synthetic data, mostly exhibiting both rotational and reflectional symmetries. Theory tells us that $\mathcal{D}_F(S)$ should produce values of zero; in practice quantization errors have caused non-symmetries, but the values remain close to zero. The fourth shape in Fig. 5 (a) has only one axis of symmetry; nevertheless, since the indentation in the square has a relatively

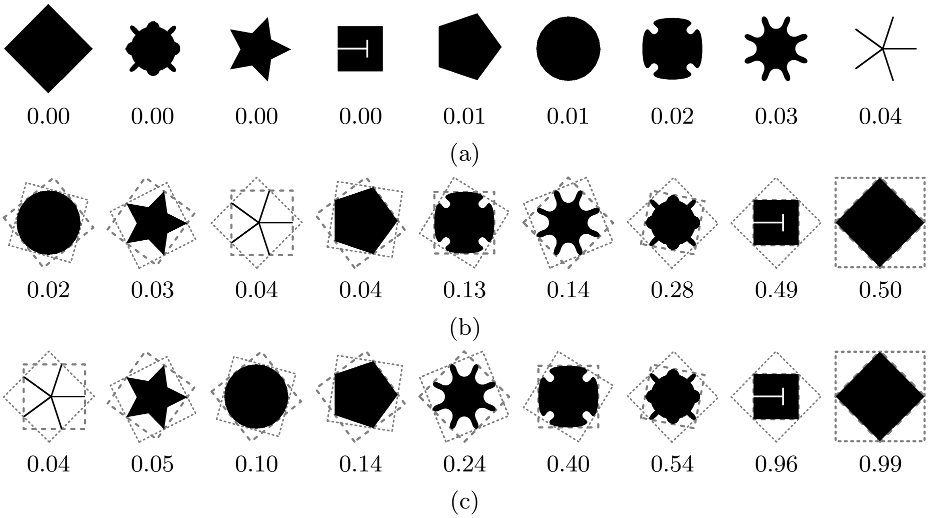

Fig. 5. Synthetic data ordered by orientability using a) $\mathcal{D}_F(S)$, b) $\mathcal{D}(S)$, c) $\mathcal{D}_{\alpha=1}(S)$. The rectangles corresponding to $\mathcal{A}_{min}$ (dashed) and $\mathcal{A}_{max}$ (dotted) are overlaid.

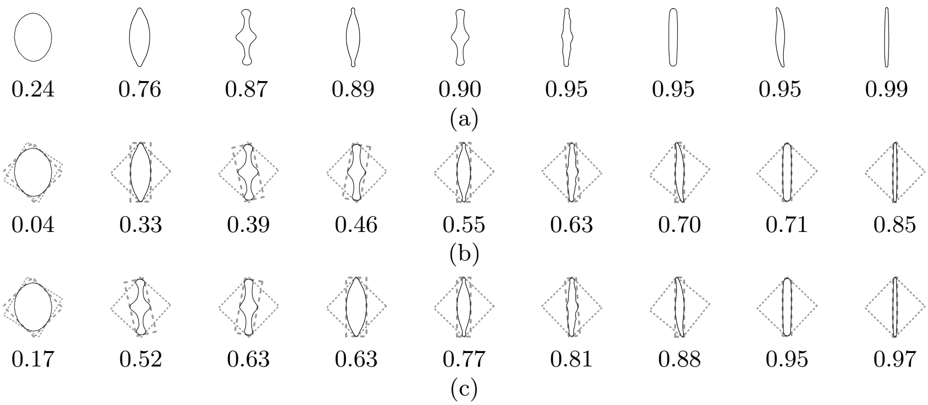

Fig. 6. Diatom data ordered by orientability using a) $\mathcal{D}_F(S)$, b) $\mathcal{D}(S)$, c) $\mathcal{D}_1(S)$

small area it does not substantially affect the values of the moments, and therefore $\mathcal{D}_F(S)$ is approximately zero. In contrast to $\mathcal{D}_F(S)$, $\mathcal{D}(S)$ does differentiate between the shapes. Again, according to theory, the first shape in Fig. 5b that looks like a circle, but is actually a 24-gon, is assigned a value close to zero.

The second set of examples (see Fig. 6) consists of the outlines of *diatoms* – unicellular water borne algae used previously by Žunić and Rosin [10] in the development of convexity measures. Future work will look at applying the orientability measure to classifying the diatoms, as in [10].

5 Concluding Remarks

We have defined shape orientability as a new shape descriptor. We also discuss some approaches for measuring shape orientability and define a new measure. The purpose of such a measure is to give an answer as to whether the computed orientation of a shape is an inherent property of the considered shape, or whether it comes from artifacts caused by the digitization process or by noise, for example. The measure can be useful if applied to shapes whose measured orientation changes even under slight deformations [1].

The shape orientability measured by the method presented here is a number in the form $[0, 1)$. The minimal possible measured orientability (equal to zero) is for a disc. There is no shape with a measured orientability equal to 1. Even in cases where there is no doubt what the orientation should be, e.g. an elongated rectangle, the measured orientability is not 1. That could be desirable property because the measured orientation for rectangles increases if the ratio between length a of the longer edge and the length b of the shorter edge increases as well. In the limit case when a is a positive constant while $b \to 0$, the measured orientability tends to 1 and we could say that a straight line segment is a perfectly oriented shape. Another desirable property is that the shapes with several axes of symmetry could have non-zero measured orientability. As an illustration, a regular $4n$-gon P_{4n} has the measured orientability $\mathcal{D}(P_{4n})$ equal to $\mathcal{D}(P_{4n}) = 1 - \frac{4\cos\frac{\pi}{4n}}{4} = 1 - \cos\frac{\pi}{4n}$. Obviously, $\mathcal{D}(P_{4n})$ tends to 0 as $n \to \infty$.

References

1. J. Cortadellas, J. Amat, F. de la Torre, "Robust Normalization of Silhouettes for Recognition Application," *Patt. Rec. Lett.,* Vol. 25, pp. 591-601, 2004.
2. H. Freeman, R. Shapira, "Determining the Minimum-Area Encasing Rectangle for an Arbitrary Closed Curve," *Comm. of the ACM,* Vol. 18, pp. 409-413, 1975.
3. B. K. P. Horn, *Robot Vision,* MIT Press, 1986.
4. R. Jain, R. Kasturi, B. G. Schunck, *Machine Vision,* McGraw-Hill, 1995.
5. R. Klette, J. Žunić, "Digital approximation of moments of convex regions," *Graphical Models and Image Processing,* Vol. 61, pp. 274-298, 1999.
6. S.E. Palmer, *Vision Science: Photons to Phenomenology,* MIT Press, 1999.
7. F.P. Preparata and M.I. Shamos, *Computational Geometry,* Springer-Verlag, 1985.
8. G.T. Toussaint, "Solving geometric problems with the rotating calipers," *Proc. IEEE MELECON '83*, pages A10.02/1–4, 1983.
9. W.H. Tsai, S.L. Chou, "Detection of Generalized Principal Axes in Rotationally Symmetric Shapes," *Patt. Rec.,* Vol. 24, pp. 95-104, 1991.
10. J. Žunić and Rosin, P.L., "A New Convexity Measurement for Polygons," *IEEE Trans. PAMI,* Vol. 26, No. 7, pp. 923–934, 2004.