Evaluating Harker and O'Leary's Distance Approximation for Ellipse Fitting

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Abstract

Harker and O'Leary's [3] recently proposed a new distance measure for conics. This paper compares its accuracy and effectiveness against several other error of fits (EOFs) for ellipses using: 1/ visualisations of the distortions with respect to the Euclidean distance; 2/ a set of evaluation measures specifically designed for assessing ellipse EOFs [7, 8]; 3/ the accuracy of LMedS ellipse fitting using the various EOFs.

1 Introduction

Conic fitting, and in particular ellipse fitting, has been found to be an extremely useful tool in computer vision, with many applications such as: face detection [16], gaze determination [17], camera calibration [4], shape measurement [11], and the analysis of grain [15], potatoes [18], sperm heads [6], etc.

There are two strands of research in ellipse fitting. The first considers alternative frameworks for performing the fitting, e.g. applying robust statistics to reduce the effects of outliers [10, 12], the use of various optimisation tools (e.g. genetic algorithms [13]), constraining the fitted conic to be an ellipse [2], constrained multiple fits (e.g. concentric ellipses) [5], and so on. The second strand focuses on the objective function, i.e. the error of fit (EOF); since the true Euclidean shortest distance between a point and an ellipse requires solving a quartic equation this is usually replaced by simpler and more efficient approximations. The most common is the so-called *algebraic distance*. Given the implicit equation of a conic

$$Q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$

then the algebraic distance from the point $\mathbf{P}_i = (x_i, y_i)$ to the conic is defined directly from the above as

$$\mathrm{EOF}_1 = Q(x_i, y_i) = ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f,$$

An often used refinement of the algebraic distance suggested by Sampson [14] is to inversely weight it by its gradient

$$\mathrm{EOF}_2 = \frac{Q(x_i, y_i)}{|\nabla Q(x_i, y_i)|}.$$

In fact, many distance approximations have been developed in recent years; thirteen are described and compared in [7, 8]. See also [1] for experimental comparison between various distance measures and fitting constraints.

Recently, Sampson's approach was revisited by Harker and O'Leary [3] who noted that Sampson provided the *distance to the first order approximation of the Euclidean distance*, while they developed the *first order approximation to the distance function.* This results in a more complicated expression which, when simplified (without loss of generality) by translating the conic such that $\mathbf{P}_i$ lies at the origin, yields

$$\mathrm{EOF}_{HO} = \sqrt{\left| -\frac{w_n}{w_d} f^2 \right|}$$

where

$$\begin{aligned} w_n &= d^4 + e^4 - 32acf^2 + 16a^2f^2 + 16b^2f^2 + 16c^2f^2 - 16bdef + 2d^2e^2 + 8cd^2f + 8ae^2f - 8ad^2f - 8ce^2f \\ w_d &= d^6 - e^6 - 3d^4e^2 - 3d^2e^4 + 10ad^4f + 10ce^4f - 8cd^4f - 8ae^4f + 18bd^3ef + 18bde^3f + \\ &\quad 32a^3f^3 + 32c^3f^3 + 32ab^2f^3 - 32ac^2f^3 - 32a^2cf^3 + 32b^2cf^3 - 32a^2d^2f^2 - 32c^2e^2f^2 - \\ &\quad 20b^2e^2f^2 - 20b^2d^2f^2 + 40acd^2f^2 + 40ace^2f^2 - 8c^2d^2f^2 - 8a^2e^2f^2 + 2ad^2e^2f + 2cd^2e^2f - \\ &\quad 24abdef^2 - 24bcdef^2. \end{aligned}$$

Although not explicitly stated in [3], the $-\frac{w_n}{w_d}f^2$ term is not always non-negative.

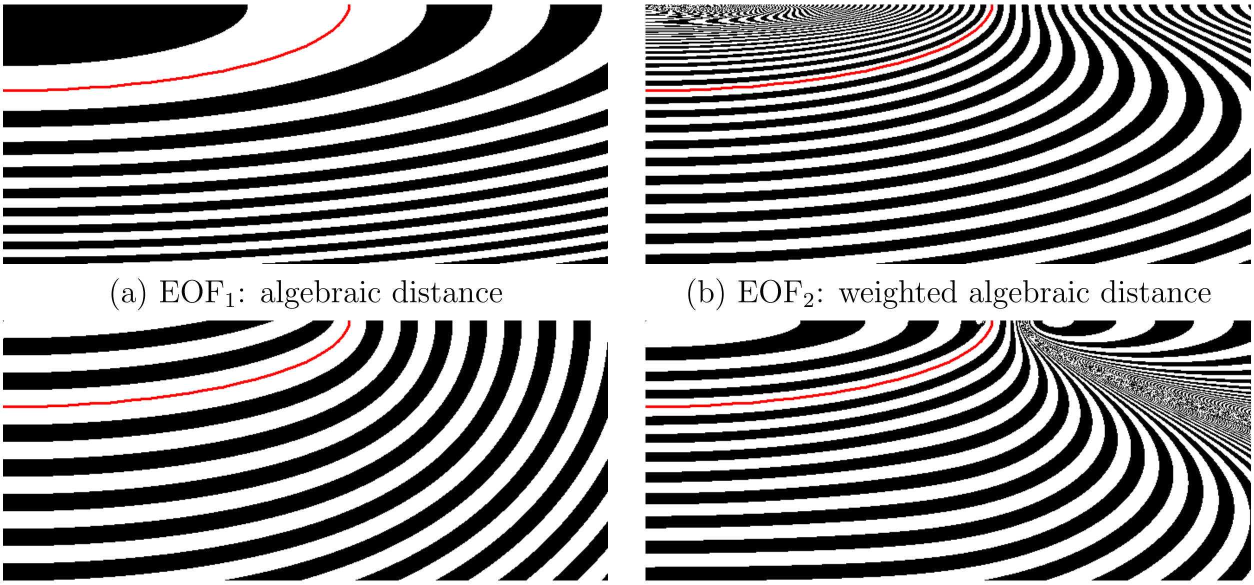
In their paper Harker and O'Leary [3] compared their distance approximation against Sampson's gradient weighted algebraic distance. However, since there are many other distance approximations available, some of which are both relatively simple and accurate, it is of interest to compare their measure against some of these.

In particular, we compare their method to one which was based on the ellipse and its confocal hyperbola that passes through $\mathbf{P}_i$ [9]. Such confocal conics are mutually orthogonal, and since much of the hyperbola is relatively straight it is a good approximation to the normal from the point to the ellipse. Computing the distance is then straightforward, and details are provided in [9].

Previously, comparisons were carried out on the above confocal conic error of fit (denoted by EOF_{14}) and the thirteen described in [7, 8], and the former (EOF_{14}) was shown to perform the best. In the same manner, the analysis is shown here for the new distance EOF_{HO} and is duplicated for EOF_1, EOF_2, and EOF_{14} for easy reference.

First, to visualise distortions in the approximate distance function, blocks of values over a fixed range are alternately coloured black and white; see figure 1. A more elongated ellipse than in [3] is used, as this makes the distortions more apparent. Although EOF_{HO} is well behaved near the ellipse it can be seen to display substantial distortions at greater distances, particularly outside the ellipse (as compared to Sampson's EOF_2 which has severe distortions inside the ellipse). In comparison, the confocal conic distance has no singularities, and is overall better behaved.

Next, a more quantitative analysis was carried out, using the methods specifically designed in [7, 8] for assessing error of fit functions for ellipses. Four basic terms were defined, quantifying the linearity of a distance approximation w.r.t. the Euclidean distance (computed by the Pearson correlation coefficient), curvature bias (how the distances vary as a function of the ellipse's curvature), asymmetry (between distances inside and outside the ellipse), and the overall goodness (denoted by G) of a distance approximation w.r.t. the



(a) EOF_1: algebraic distance

(b) EOF_2: weighted algebraic distance

(c) EOF_{14}: confocal conic distance

(d) EOF_{HO}: Harker and O'Leary's distance

Figure 1: Iso-distance bars of distance approximations to the ellipse (in red) which has aspect ratio = 4.

Euclidean distance. The overall goodness is computed on all distances from the ellipse, weighted by a drop-off function. Here we have used a Normal distribution, and so its standard deviation determines whether goodness is primarily measured according to distance values close to the ellipse (for a low standard deviation) or should also substantially include more distant values (for larger standard deviations). See [7, 8] for more details.

To facilitate comparison, all values have been normalised with respect to the algebraic distance. At small distances from the ellipse, table 1 shows that the weighted algebraic distance is generally better than the algebraic distance except for its severe asymmetry between inside and outside. Both EOF_{14} and EOF_{HO} offer comparable improvements in overall goodness, while EOF_{HO} is especially superior in asymmetry but not as effective as EOF_{14} regarding curvature bias. However, at larger distances from the ellipse the distortions of Harker and O'Leary's distance become more apparent, as quantified in table 2.

EOF	L	C	A	G
1	1.000	1.000	1.000	1.000
2	0.987	0.011	3.980	0.808
14	1.000	0.009	1.107	0.775
HO	1.000	0.027	0.469	0.774

Table 1: Normalised assessment results with $N(0, 2)$ noise model; aspect ratio = 4

Finally, we test the quality of the distance approximations by fitting ellipses using the Least Median of Squares (LMedS) approach [10]. 32000 sets of synthetic data were generated for ellipses, each containing between 10 and 26 points, varying the following parameters:

EOF	L	C	A	G
1	1.000	1.000	1.000	1.000
2	0.877	0.041	8.404	2.771
14	1.006	0.000	0.755	0.002
HO	1.002	0.275	0.784	2.978

Table 2: Normalised assessment results with $N(0, 64)$ noise model; aspect ratio = 4

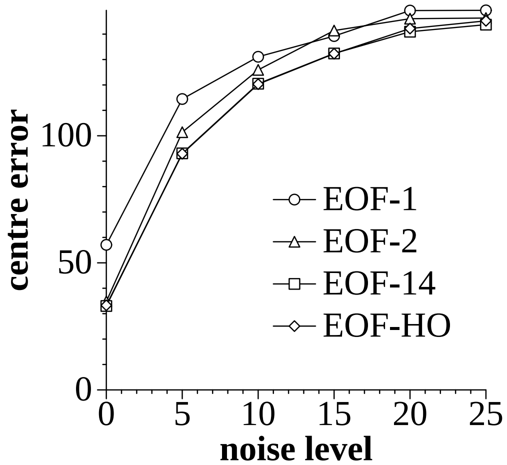


Figure 2: Errors in estimated centre position for 32000 synthetic ellipses

major axis $a \in [200, 450]$, minor axis $b = 100$, subtended angles $\theta \in [1, 2.5]$ radians, and added Gaussian noise $\sigma \in [0, 40]$. The alpha trimmed mean[1] ($\alpha = 0.1$) errors of the centre estimate are plotted in figure 2. First, note that even for zero added noise there is quite a high error due to the effect of quantisation (all data points were rounded to integer) which can have a significant impact on the fits to sections with short subtended angles. The alpha trimmed mean is used since all the methods occasionally generate totally erroneous fits with huge errors ($> 10^7$), corrupting their non-trimmed mean errors. Second, we see that there is little difference between EOF_{14} and EOF_{HO}, and that both of them perform better than the other methods. The similarity between EOF_{14} and EOF_{HO} may seem surprising since the latter was shown to display substantial distortion (unlike EOF_{14}). However, it is localised to a relatively small area, and most noticeable at high ellipse aspect ratios and noise levels, which is why it has not more seriously adversely affected its effectiveness in the fitting experiment.

[1]The α-trimmed mean is calculated by sorting the data, removing the top and bottom $\alpha \times 100\%$ of the data, and calculating the mean of the remaining data.

2 Conclusions

Harker and O'Leary's [3] recently proposed a new distance measure for conics. This paper shows that while their method does indeed generally outperform the inverse gradient weighted algebraic (i.e. Sampson's) distance it is not superior to Rosin's confocal conic method [9].

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