The Amphitheatre Construction Problem

Paul L. Rosin
Cardiff School of Computer Science
Cardiff University
Cardiff, CF24 3AA, UK
`Paul.Rosin@cs.cf.ac.uk`

Emanuele Trucco
School of Engineering and Physical Sciences
Heriot-Watt University
Edinburgh, EH14 4AS, UK
`e.trucco@hw.ac.uk`

While stands the Coliseum, Rome shall stand;
When falls the Coliseum, Rome shall fall;
And when Rome falls – the world.

Childe Harold's Pilgrimage, Canto iv. Stanza 145.
Lord Byron

Abstract

In this paper we look at the problems of designing and laying out amphitheatres. While they are obviously oval there has been much argument as to the precise geometry adopted for their construction. One possibility is that they are elliptical. Another is that, to simplify certain issues, they were designed as ovals, made up as piecewise circular approximations to ellipses. This would fit in with the Romans' practical nature.

This paper examines the problem of the layout of the Colosseum from a statistical point of view. We fit the curves generated by the models (the true ellipse and various ovals) to a set of planimetric points acquired from a survey of the Colosseum. We then carry out a detailed residual analysis in order to decide which curve yields the best fit.

1 Introduction

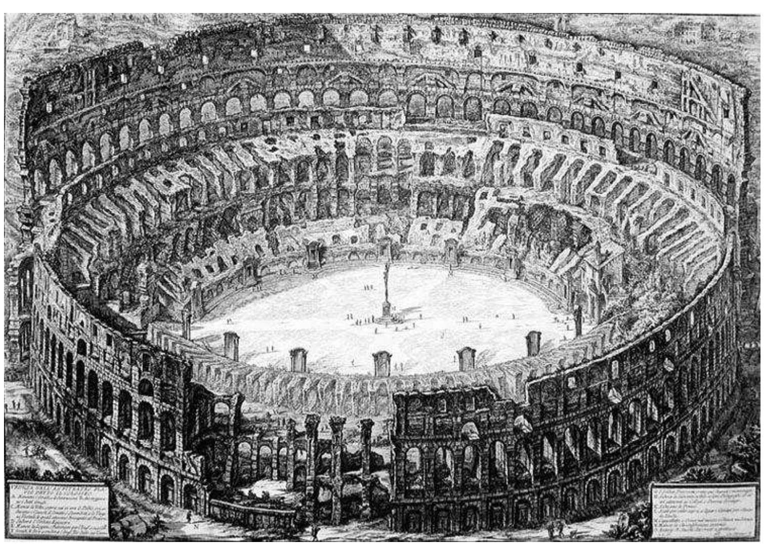


Figure 1: View of the Colosseum, G.B. Piranesi, circa 1760

In this paper we look at the problems of designing and laying out amphitheatres. Such buildings were new to the Romans, not occurring in Hellenic times, but became a standard part of the Roman city, hundreds being built. Golvin [12] argues that in the early days they were built in the traditional rectangular shape, but evolved over the years, becoming oval. This was in response to several factors. The first (Figure 2a) is to avoid corners and flat sides, which would otherwise reduce the scope for action in the arena. Second, seating all round the oval amphitheatre would have reasonable views, unlike seats in the far corners of a rectangular amphitheatre (Figure 2b).

While amphitheatres are obviously *oval* (using that term in a loose sense) there has been much argument as to their precise shape. Unfortunately, there is little relevant contemporary writing on architecture and construction, and even that does not provide any firm evidence (e.g. Vitruvius [30], Balbus[1] [4]).

One possibility is that they are elliptical. It is conceivable that the peg-and-rope (gardener's method) technique might have been used to lay out an ellipse. However, while adequate for drawing rough, small ellipses there are several difficulties in scaling the approach up to tackle large ellipses. Moreover, accuracy is compromised by factors such as sag and elasticity in the rope, variations in tension, humidity, and temperature, etc. Other means to construct ellipses are possible, e.g. the envelope of lines generated from a rectangle, or a trammel, but they are also prone to problems. A secondary issue with an elliptical design, which we will come back to, is the drawing of parallels to the ellipse for the rows of seating.

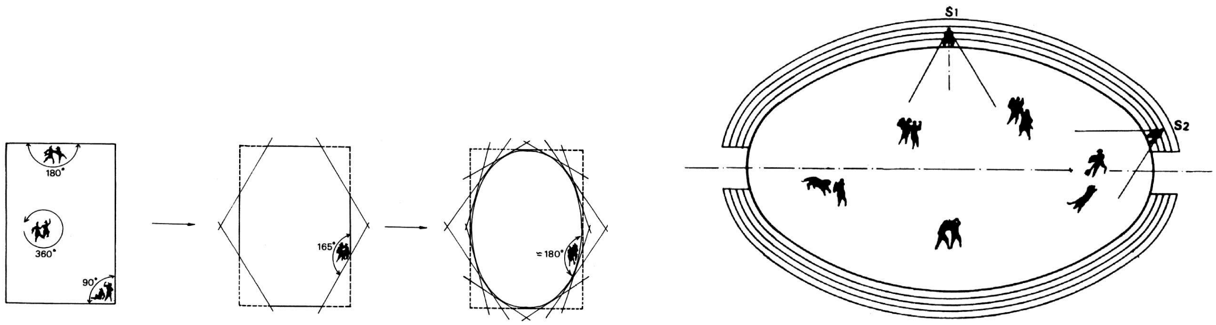
Alternatively, in order to simplify such issues, the amphitheatres may have been designed using ovals which were piecewise circular approximations to ellipses. This would fit in with the Romans' practical nature. For instance, Vitruvius approximates the spiral volute on the capital of an Ionic column by joining quarter circles of decreasing radii whose centres move around a small central square. Although the equation for a spiral would have been well understood (e.g. Archimedes' "On Spirals"), and it could have been drawn out using a peg and string, circular arcs are much easier to handle. In a similar vein, Wilson Jones argues that the subtle swelling of columns (entasis) could have been shaped by the Romans using straight lines rather than more complicated curves such as conics or conchoids [17]. Another example of Roman practicality, this time in the efficient use of materials, is shown in the clever use of irregular stone blocks for building walls. By incising grooves to form pseudo-joins between blocks the appearance of regularity was obtained [1]. Finally, it it noteworthy that while in architecture the classical Greek mouldings are derived from conic sections, Roman mouldings are all based on circular arcs.

Working with ovals has several advantages: they are easier to lay out (and therefore less error-prone), and parallels can be easily and accurately drawn. Even if an oval was used, there still remains the question of which one. There are many procedures for drawing ovals that were published from the Renaissance [26] through to the twentieth century [25].

In contrast to the paucity of original writings on the subject there is a large literature on how the Roman amphitheatres were designed and laid out. Many authors propose ovals constructed from circles in one form or another (also known as polycentric arcs, or quadrarcs, four centered ovals, eight centered ovals, etc.). Examples, ranging over the last 200-300 years include: Fontana [10], Maffei [19], Devecchi [7], Guadet [14], Cozzo [5], and Golvin [12]. Others however put the case forward for the ellipse: Biradi, [3], Michetti [20], Trevisan [29], de Rubertis [6], and Sciacchitano [27]. It is also possible that different schemes were used for different amphitheatres, for instance an ellipse for the small relatively crude amphitheatre at Pompeii, and polycentric ovals at most of the others [16, 9].

Several authors note that the convergence of the axes of the radial partition walls to four points supports the hypothesis of the four centred oval construction [8, 22]. Nevertheless, the ellipse gives a good fit [29], and maybe indicates that separate phases for the design and construction took place [27]. Ultimately, the differences between the ellipse and oval models are subtle, making it difficult to determine which (if any) is the true underlying shape. In this paper we aim to provide a more objective and valid method of assessment, based on a statistical point of view. While the methodology is applicable to any amphitheatre, in this paper we have applied our analysis to the Colosseum (Figure 1).

[1] While Wilson Jones [16] reads the phrase *harenae ex quattuor circulis* in Balbus as indicating a four arc oval, Campbell [4] interprets it rather differently as four concentric circles [4].



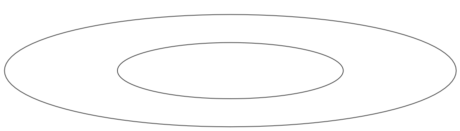
(a) (b)

Figure 2: Illustrations from Golvin [12]; a) the genesis of the elliptical shape of amphitheatres, b) good views obtained all round the amphitheatre.

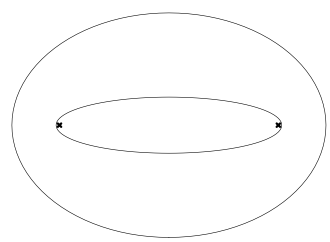
2 Difficulties with Parallels to Ellipses

The laying out of the rows of seating in the amphitheatre would be facilitated by a means of drawing parallels to the arena. Some authors have assumed that this could be done by generating a set of parallel ellipses (e.g. Golvin [12], Wilson Jones [16]). However, this is incorrect since the parallels of an ellipse are not ellipses themselves, but are actually eighth order polynomials containing 104 coefficients [13]!

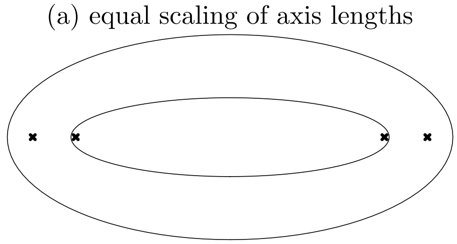
Figure 3 illustrates some of the difficulties in approximating a parallel to an ellipse using another ellipse. To accentuate the errors we have used a fairly elongated ellipse (major axis $a = 200$ and minor axis $b = 50$). Three attempts at creating outer parallels are shown. The first simply scales both the major and minor axes lengths by the same amount, keeping the eccentricity constant. It can be seen that the parallel bulges out at the pointed end of the ellipse. The second retains the two foci of the initial ellipse. If the gardener's method for drawing an ellipse is used this would be the simplest approach to generating parallels as the string would be lengthened but the pegs do not need to be moved. However, it has the opposite defect to uniform scaling: the pointed ends are squashed in. The third method increases both the axis lengths by the same offset. This effectively also shifts the foci outwards from the centre of the ellipse. It can be seen to perform much better than first two naïve methods. In fact the errors are fairly subtle, but are evident when the true parallel is overlaid (Figure 3d). The parallel is squarer in shape than the elliptical approximation.



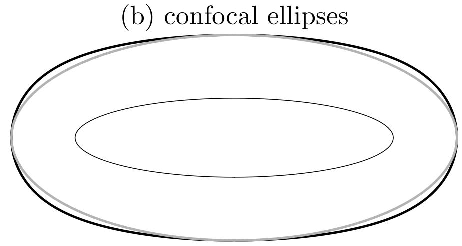
(a) equal scaling of axis lengths



(b) confocal ellipses



(c) equal increase in axis lengths



(d) true parallel to ellipse overlaid on (c)

Figure 3: Different types of concentric ellipses

The true parallel calculated for comparison in Figure 3d was generated using a simpler approach than working with the eighth order polynomial description of the ellipse parallel in implicit form. This is done by plotting points $(x + x_d, y + y_d)$ at distance d along the normal to the ellipse at (x, y). For any parametric equation $\{x(t), y(t)\}$ this can be calculated as

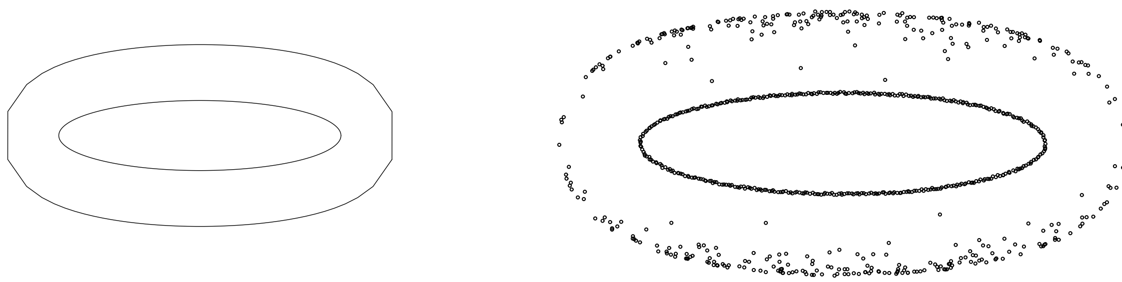
$$\{x_d, y_d\} = \frac{d}{\sqrt{x'(t)^2 + y'(t)^2}} \{y'(t), -x'(t)\}$$

and so for an ellipse

$$\{x(\theta), y(\theta)\} = \{a\cos\theta, b\sin\theta\}$$

we obtain

$$\{x_d, y_d\} = \frac{d}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \{b\cos\theta, a\sin\theta\}.$$



(a) using 80 points evenly distributed along the ellipse (b) using 400 noisy points

Figure 4: Applying Ashpitel's method for forming perpendiculars to ellipses

Discounting the above approaches as inaccurate or impractical, constructing fixed length perpendiculars to the ellipse seems like a good candidate for finding the parallels to an ellipse. We know of no reference to any technique in Roman literature, but it is possible they used similar methods to more recent builders.

Ashpitel [2] gives two approaches. The first relies on the focal property of the ellipse that the two lines from a point on the ellipse to the foci make equal angles with the tangent to the ellipse. Thus the normal to the ellipse is the angular bisector of these two lines. This property is described by Apollonius ("On Conics", Book III, propositions 47-48) and so was probably known to the Romans. However, theoretically possible, it would probably be tricky to carry out.

The second method is simpler and more practical, although it only provides an approximate solution. The procedure to find the perpendicular from point p on the ellipse is to draw out circles centred at two other points which are evenly spaced along the ellipse on either side of p. The line through the two points of intersection gives the perpendicular. It can be seen that the method works perfectly if the curve section between the pair of points is symmetric, e.g. a straight line or a circle. Otherwise only an approximation to the perpendicular is obtained. An additional problem besides the inherent inaccuracy of the approximation is in making an appropriate choice of the distance between the pair of points. If the points are too far apart this will lead to inaccuracies as the asymmetry becomes more apparent. On the other hand, if the points are too close then the technique becomes very sensitive to errors in the positions of the points. Figure 4a shows the results obtained when eighty points along the ellipse are used. The inaccuracies at the points of high curvature are evident. When more points are used the results on this synthetic data are almost perfect. However when even a slight amount of noise is added to these more closely spaced points (uniformly distributed in the range $[-1, 1]$ units where the axes lengths are $a = 200$ and $b = 50$ as before) the perpendiculars become unreliable. The intersections exterior to the ellipse are marked in Figure 4b. Note that the estimates are biased. Whereas overestimates are small and tend not to exceed the radius being drawn out from the two points, underestimates are considerably larger.

3 Serlio's Oval Constructions

There are many possible piecewise circular constructions of ovals [25, 26]. We shall concentrate on the first published method, given by Serlio [15], whose four methods of construction are illustrated in Figure 5; these have used extensively in architecture over the years.

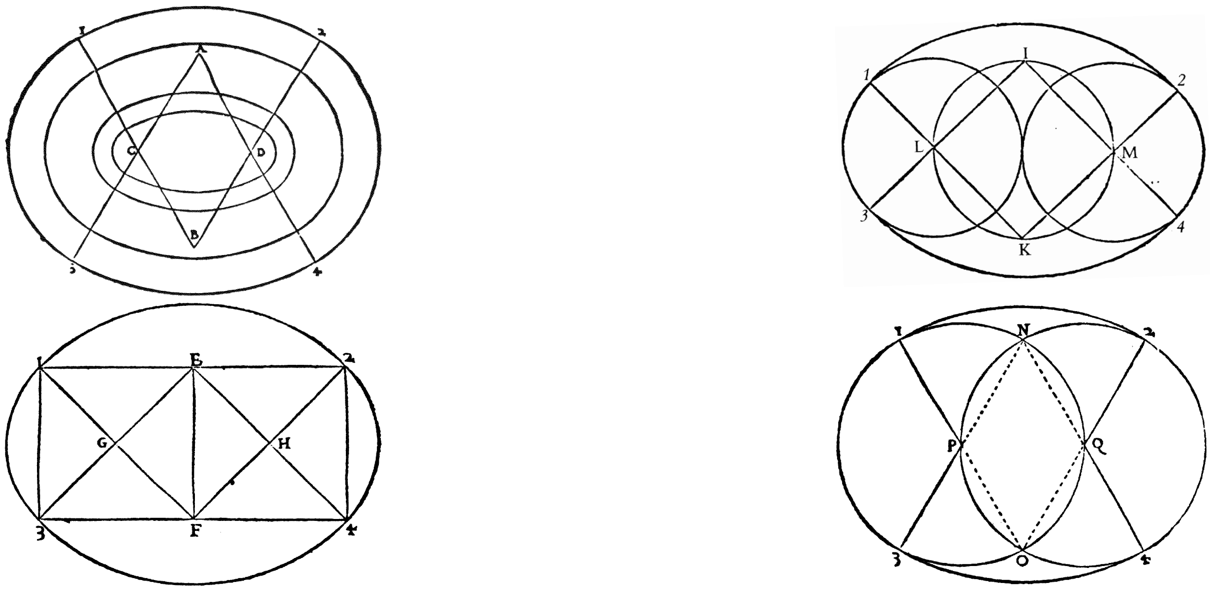


Figure 5: Sebastiano Serlio's oval constructions (1537–1575).

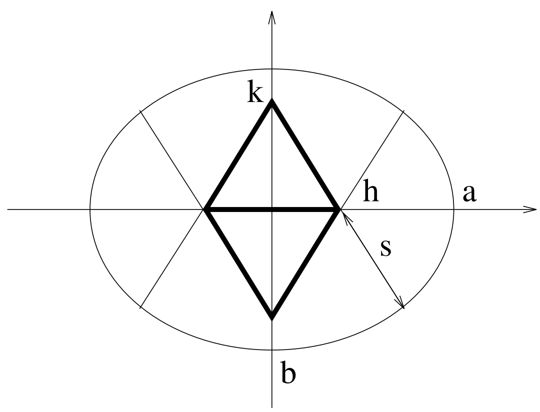


Figure 6: Serlio's construction method for variable aspect ratio ovals

Only the first construction allows a variable aspect ratio; the other three are all fixed. It is the former we shall use in the following analysis. The oval consists of four circular arcs with centres $(\pm h, 0)$ and $(0, \pm k)$ and radii $a - h$ and $b + k$ respectively (see Figure 6). It has the attractive property that the arcs join smoothly with tangent continuity. This geometric constraint can be algebraically expressed as

$$h = \frac{k - \frac{a-b}{2}}{\frac{k}{a-b} - 1}. \tag{1}$$

The construction can be generated conveniently using two equilateral triangles whose bases are centred on the origin. Their intersections with the axes determine h and k, which can be expressed as

$$h = \frac{a-b}{\sqrt{3}-1}; \quad k = \frac{\sqrt{3}(a-b)}{\sqrt{3}-1}.$$

The lengths of the circular arcs are specified by extending the triangles' diagonal sides by a length s. When s is increased both a and b are increased, and so it is not straightforward to (geometrically) choose correct values of h and s so as to achieve specified values of a and b. In fact, the ratio is given by

$$\frac{a}{b} = \frac{h+s}{h(2-\sqrt{3})+s}.$$

Not all aspect ratios can be achieved with this method since two arcs per quadrant are only drawn when $s > 0$ such that $\frac{a}{b} \leq \frac{1}{2-\sqrt{3}} \approx 3.732$. The ratio of the radii of the two circular arcs also varies, and can be shown to be $\frac{s}{2h+s}$.

4 Model Fitting to Noisy Data

The majority of the papers in the literature analysing the layout of amphitheatres tend to overlay (often manually) the proposed shape (ellipse or oval) and then visually evaluate its appropriateness. While this might be feasible with perfect data the Colosseum has been subjected to substantial and prolonged damage over the years, leaving it in a ruinous state. Not only have there been natural hazards such as erosion, ground subsidence, earthquakes, fire, and lightning [11], but also many alterations and repairs have been carried out [18]. On top of this, it is quite natural that inaccuracies and local adjustments would occur in construction. Thus, the positions of the walls no longer perfectly correspond to either an ellipse or a piecewise circular oval.

This problem is aggravated by the similarity between an ellipse and an oval. Depending on the number of circular arcs used an oval can be found that is a very good approximation of an ellipse [25]. In the context of the Colosseum it has been noted that the difference between a four centred oval and an ellipse only amounts to a few centimetres [8, 21]. Given such subtle differences, visual evaluation is inadequate, and a more objective approach is required. This has led researchers to perform a fit between the proposed models (i.e., ellipse, polycentric ovals) and use the fitting error as a criterion for selecting the most appropriate model [29].

In this paper we also perform a best fit between the candidate models and the data. Rather than use a traditional least squares error term we minimise the mean absolute error using Powell's method [23]. This is less sensitive to outliers than least squares. In the future other, yet more robust, methods will be considered [24]. The error at each data point is taken as the shortest distance to the oval or ellipse. For the former the appropriate arc needs to be selected, making the fitting a non-linear, iterative procedure. In the case of the ellipse, computing the distances requires solving a quartic equation and choosing the shortest of the four solutions, and so it is also a non-linear process. In addition to Serlio's oval, the best fitting four centred oval that satisfies the tangent continuity constraint (Equation 1) is found.

5 Colosseum Data

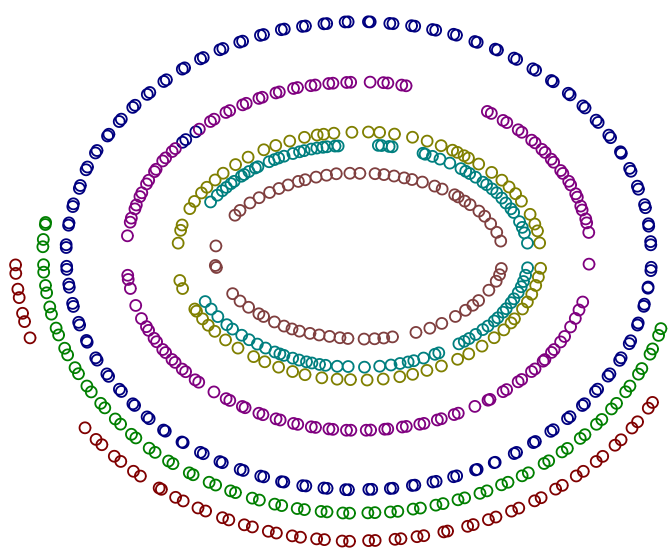


Figure 7: Measured data from the Colosseum

Figure 7 shows the data acquired from the Colosseum; there are 662 points in total, arranged in seven rings, which we have labelled 1-7 going from the outermost to the innermost. The ellipse and the two oval models were fitted to the data as described above, and the fits to two of the rings are illustrated in Figure 8. It can be seen that all curves provide a very good fit to the data.

ring	1	2	3	4	5	6	7
ellipse	**0.036**	**0.035**	**0.060**	**0.061**	**0.089**	**0.105**	**0.118**
Serlio's oval	0.183	0.146	0.365	0.284	0.275	0.239	0.205
optimal oval	0.133	0.128	0.159	0.134	0.137	0.129	0.131

Table 1: Mean absolute errors of fitted models to the Colosseum data

The mean absolute residual errors are given in Table 1. The best fits (lowest errors) are highlighted and it can be seen that, in accordance to Trevisan's [29] findings, the ellipse provides the best match to the data. However, despite the error values providing such objective and quantitative information, interpretation of these results must still be done with care. It is not sufficient to identify the best fitting model and declare that this was the one used in the planning and construction of the amphitheatre. For any set of data one of the models is bound to give a lower error than the remaining models. However, the question is whether that improvement in error of fit between one model and another is *statistically* significant.

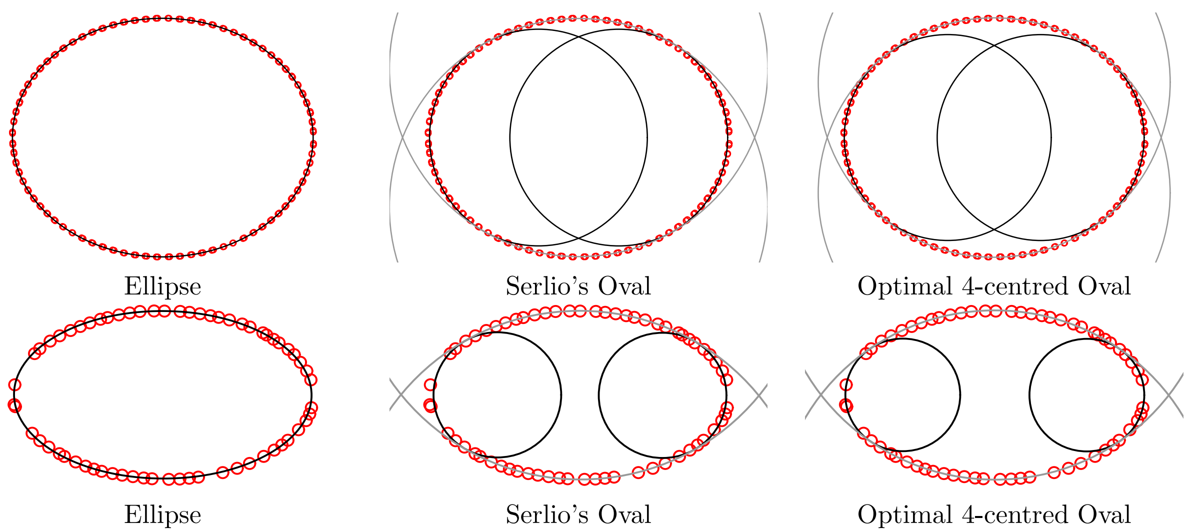


Figure 8: Best fit curves overlaid on the Colosseum data (rings 3 and 7)

6 Statistical Analysis

To analyse the data the first step is to determine what statistical model is appropriate. The fact that the different models have different degrees of freedom makes directly comparing them based on their mean fitting errors problematic. Instead we shall compare the models via their distributions of errors, and test if two error distributions are significantly different or not.

The simplest and most common model assumes normal distributions. Histograms of the *signed* residual errors for two of the rings are given in Figure 9. While some histograms look reasonably normal others are doubtful, moreover their appearance is somewhat dependent on the histogram bin width. To check the hypothesis of normality more thoroughly the Shapiro-Wilk test [28] is applied to the signed residuals (Table 2). If $p < 0.05$ then the null hypothesis of normality is rejected, and so only a small number of fits (indicated by the highlighted values) can be considered to have normally distributed residuals.

The above means that many standard statistical tests for comparing hypotheses are not applicable to this data. A non-parametric test is more appropriate since it does not require the assumption of a normal distribution. However, even when considering non-parametric tests, most still make some assumptions

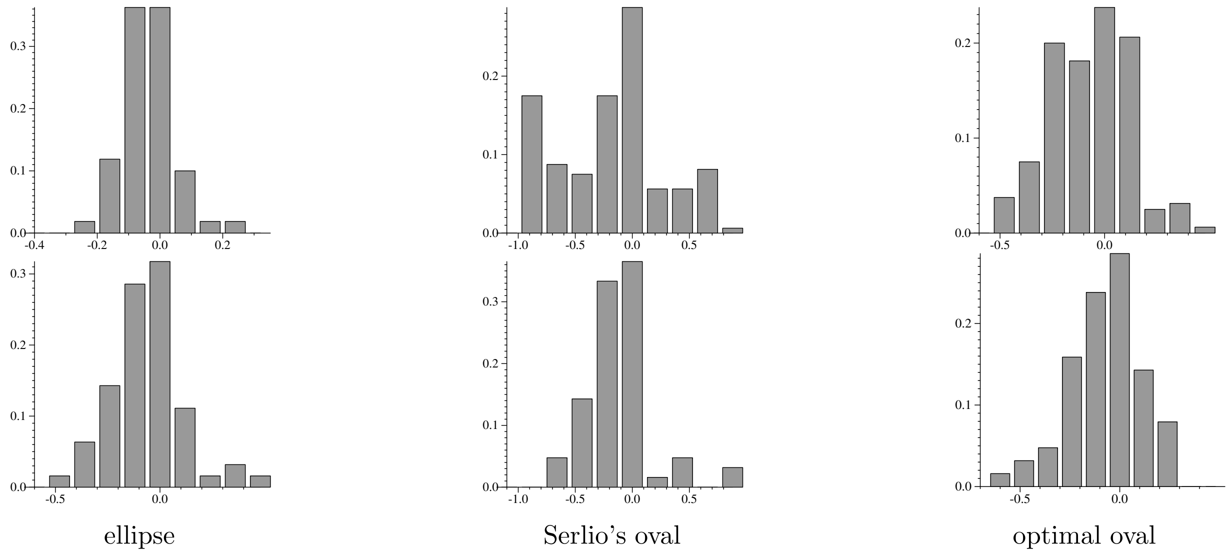


Figure 9: Histograms of residuals of fitted curves to two sets of data points (rings 3 and 7) from the Colosseum

ring	1	2	3	4	5	6	7
ellipse	**0.4060**	**0.1013**	0.0011	0.0059	**0.4650**	0.0000	0.0032
Serlio's oval	0.0001	0.0001	0.0000	0.0003	0.0211	0.0000	0.0000
optimal oval	**0.2250**	0.0385	**0.2411**	0.0474	**0.1663**	**0.2811**	**0.3408**

Table 2: Normal test computed by the Shapiro-Wilk statistic; only the highlighted values can be considered normal

on the two groups of data (error distributions in our case) such as i) they have the same standard deviations, ii) they have symmetric distributions, iii) the distributions have the same shape. There are few tests that do not assume any of the above, and so we are forced to use the median test, which tests if two independent groups differ in central tendencies (the alternative hypothesis) [28]. The limitation of the median test is that since it makes so few assumptions it inevitably has a low statistical power, i.e., it needs more data points and/or lower noise levels to reach similar confidence levels than other comparable tests that make more assumptions (especially parametric tests).

ring	1	2	3	4	5	6	7
Serlio's oval	3.705	0.547	**47.535**	**15.353**	**7.055**	2.617	0.942
optimal oval	**14.325**	4.369	**30.948**	**17.811**	1.576	1.268	0.145

Table 3: χ^2 values computed by the median test and applied to detect significant difference between the residuals from the ellipse fit and either of the two oval fits; only the highlighted values can be considered statistically significantly different

The results of the median test are shown in Table 3. If $\chi^2 \geq 6.64$ then the null hypothesis of no difference between the distributions (at a significance level $\alpha = 0.01$) is rejected. In the majority of the cases there was insufficient information in the data to allow the median test to discriminate between the ellipse and ovals. This implies that although the ellipse model uniformly fitted the data better than the four centred ovals, it is not possible to prove that this is statistically significant.

7 Conclusions

In this paper we have made an analysis of ellipse and four centred oval fits to the concentric walls of the Colosseum. We have shown that both models provide accurate matches to the data, but the ellipse always fits better. However, these differences are relatively small and subtle, and to determine if they are significant, the median test was applied to compare the distributions of residuals.

It was found that the median test was unable to identify a statistically significant difference between the ellipse and the oval fits in most cases. This can be explained in several ways. First, maybe the ellipse really is the correct model, but it was not possible to verify this due to the low power of the median test and the noisy nature of the data. Second, maybe more robust techniques would have led to better fits, leading to more discriminating residuals. Third, maybe the four-centre oval is the wrong model to compare against. Other ovals (e.g. with eight and even twelve centres) have been considered in the literature. If laid out or fitted correctly they might provide better fits than the four centred oval, and therefore provide a greater challenge to the ellipse.

8 Acknowledgements

The authors wold like to thank the following for help in obtaining the Colosseum data: Profs. Riccardo Migliari, Camillo Trevisan, Mario Docci, and Renzo Carlucci.

References

[1] J.P. Adam. *Roman Building: Materials and Techniques.* B.T. Batsford, 1994.

[2] A. Ashpitel. *Treatise on Architecture, including the arts of construction, building, stone-masonry, arch, carpentry, roof, joinery, and strength of materials.* A. and C. Black, 1867.

[3] G. Birardi, R. Carlucci, E. Ferrara, U. Giannoni, and G. Maruffi. The photogrammetric survey of the Coliseum in Rome. In *International Archives of Photogrammetry, Commision V, Kyoto*, 1988.

[4] B. Campbell. *The Writings of the Roman Land Surveyors: Introduction, Text, Translation and Commentary.* Journal of Roman Studies Monographs no. 9, Society for the Promotion of Roman Studies, 2000.

[5] G. Cozzo. *L'Anfiteatro Flavio nella tecnica edilizia e nella storia delle strutture nel concetto esecutivo dei lavori.* Fratelli Palombi Editori, 1971.

[6] R. de Rubertis. Un enigma avvincente: il tracciato planimetrico ellittico del Colosseo. *Disegnare idee immagini*, (18-19):99–105, 2000.

[7] C. Devecchi. *L'Anfiteatro Flavio detto il Colosseo.* 1821.

[8] M. Docci. La forma del Colosseo: dieci anni di ricerche. il dialogo con i gromatici romani. *Disegnare idee immagini*, (18-19):23–32, 2000.

[9] S. Duvernoy. Architecture and mathematics in Roman amphitheaters. *Nexus Network Journal*, 4(3), 2002.

[10] C. Fontana. *L'Anfiteatro Flavio.* 1725.

[11] R. Funiciello, L. Lombardi, F. Marra, and M. Parotto. Seismic damage and geological heterogeneity in Rome's Colosseum area: are they related? *Annali di Geofisica*, 38(5-6):927–937, 1995.

[12] J.C. Golvin. *L'Amphitheatre Romain.* Boccard, 1988.

[13] N.T. Gridgeman. Elliptic parallels. *The Mathematics Teacher*, 63:481–485, 1970.

[14] F. Guadet. *Étude sur la construction et la disposition du Colisée.* Librairie Centrale des Beaux Arts, 1878.

[15] V. Hart and P. Hicks, editors. *Sebastiano Serlio on Architecture : Books I-V of "Tutte L'Opere D'Architettura et Prospetiva"*. Yale Univ. Press, 1996.

[16] M. Wilson Jones. Designing amphitheatres. *Mitteilungen des Deutschen Archaologischen Instituts: Römische Abteilung*, 100:391–442, 1993.

[17] M. Wilson Jones. The practicalities of entasis: Refinements of classical architecture. In *International Symposium on Classical Architecture*, pages 225–249, 1999.

[18] L.C. Lancaster. Reconstructing the restoration of the Colosseum after the fire of 217. *Journal of Roman Archaeology*, 11:146–174, 1998.

[19] S. Maffei. Degli anfiteatri e singolarmente del Veronese. In J. Poleno, editor, *Thesauri antiquitatum romanarum graecarumque*. 1737.

[20] A. Michetti. Possibili costruzioni delle ellissi del Colosseo. *Disegnare idee immagini*, (18-19):89–98, 2000.

[21] R. Migliari. Ellissi e ovali: Epilogo di un conflitto. *Palladio*, Anno VIII(16):93–102, 1995.

[22] R. Migliari. Principi teorici e prime acqisizioni nel rilievo del Colosseo. *Disegnare idee immagini*, (18-19):33–50, 2000.

[23] W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vettering. *Numerical Recipes in C*. Cambridge University Press, 1990.

[24] P.L. Rosin. Further five-point fit ellipse fitting. *CVGIP: Graphical Models and Image Processing*, 61(5):245–259, 1999.

[25] P.L. Rosin. A survey and comparison of traditional piecewise circular approximations to the ellipse. *Computer Aided Geometric Design*, 16(4):269–286, 1999.

[26] P.L. Rosin. On Serlio's constructions of ovals. *Mathematical Intelligencer*, 23(1):58–69, 2001.

[27] E. Sciacchitano. Il Colosseo. geometria dell'impianto. *Disegnare idee immagini*, (18-19):107–116, 2000.

[28] S. Siegal and N. J. Castellan. *Non-Parametric Statistics for the Behavioural Sciences*. McGraw Hill, 1988.

[29] C. Trevisan. Sullo schema geometrico costruttivo degli anfiteatri romani: gli esempi del Colosseo e dell'arena di Verona. *Disegnare idee immagini*, (18-19):117–132, 2000.

[30] Vitruvius. *De Architectura*. Dover, 1960.