CM3106 Chapter 2: DSP, Filters and the Fourier Transform

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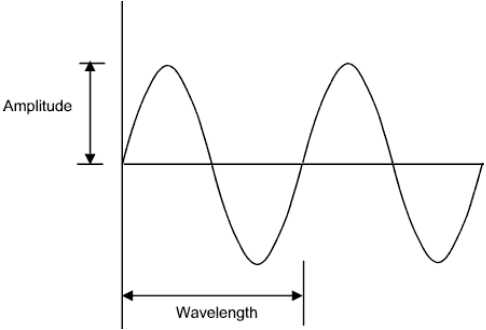
Digital Signal Processing and Digital Audio Recap from CM2104/CM2208

Issues to be Recapped:

- Basic Digital Signal Processing and Digital Audio
  - Waveforms and Sampling Theorem
  - Digital Audio Signal Processing
  - Filters

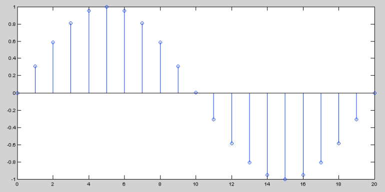
For full details please refer to last Year's **CM2208 Course Material** — **Especially detailed underpinning maths** — and also **CM2104 Notes**.

Simple Waveforms



- **Frequency** is the number of cycles per second and is measured in Hertz (Hz)
- **Wavelength** is *inversely proportional* to frequency i.e. Wavelength varies as $\frac{1}{frequency}$

The Sine Wave and Sound



The general form of the sine wave we shall use (quite a lot of) is as follows:

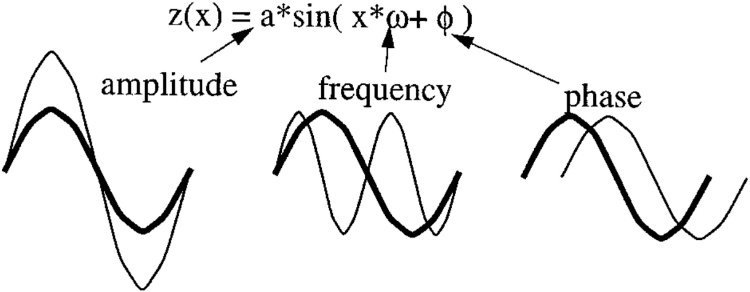
$$y = A.sin(2\pi.n.F_w/F_s)$$

where:

A is the amplitude of the wave,
F_w is the frequency of the wave,
F_s is the sample frequency,
n is the sample index.

MATLAB function: `sin()` used — works in radians

Relationship Between Amplitude, Frequency and Phase

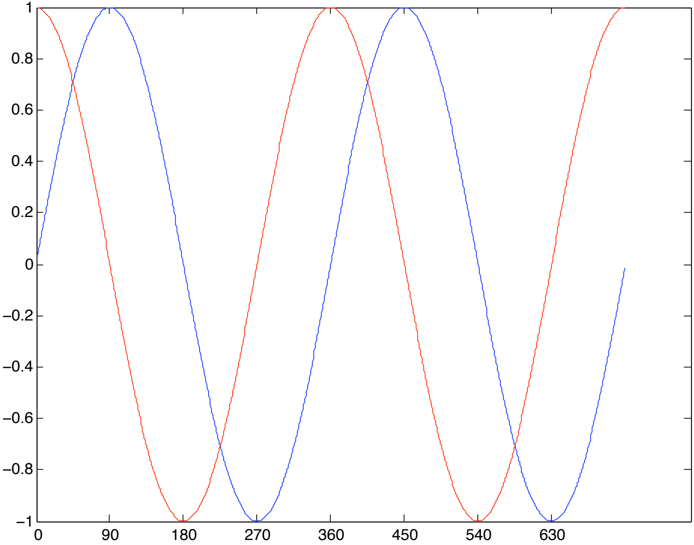


Phase of a Sine Wave

sinphasedemo.m

```
% Simple Sin Phase Demo
samp_freq = 400;
dur = 800; % 2 seconds
amp = 1; phase = 0; freq = 1;
s1 = mysin(amp,freq,phase,dur,samp_freq);

axisx = (1:dur)*360/samp_freq; % x axis in degrees
plot(axisx,s1);
set(gca,'XTick',[0:90:axisx(end)]);

fprintf('Initial Wave: \t Amplitude = ...\n', amp, freq, phase,...);

% change amplitude
phase = input('\nEnter Phase:\n\n');

s2 = mysin(amp,freq,phase,dur,samp_freq);
hold on;
plot(axisx, s2,'r');
set(gca,'XTick',[0:90:axisx(end)]);
```

Phase of a Sine Wave: `sinphasedemo` output



Basic DSP Concepts and Definitions: The Decibel (dB)

When referring to measurements of power or intensity, we express these in decibels (dB):

$$X_{dB} = 10 \log_{10} \left(\frac{X}{X_0} \right)$$

where:

- X is the actual value of the quantity being measured,
- X_0 is a specified or implied reference level,
- X_{dB} is the quantity expressed in units of decibels, relative to X_0.
- X and X_0 **must** have the same dimensions — they must measure the same type of quantity in the the same units.
- The reference level itself is **always at 0 dB** — as shown by setting $X = X_0$ (**note:** $\log_{10}(1) = 0$).

Why Use Decibel Scales?

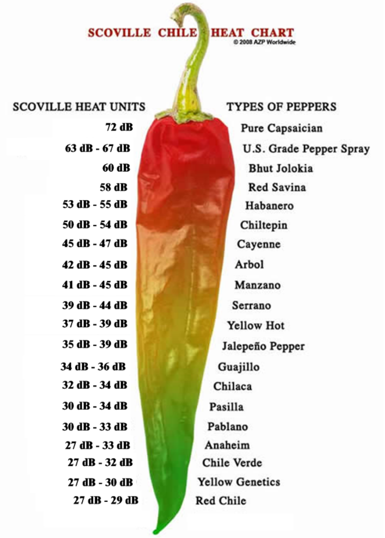
- When there is a large range in frequency or magnitude, logarithm units often used.
- If X is greater than X_0 then X_{dB} is positive (Power Increase)
- If X is less than X_0 then X_{dB} is negative (Power decrease).
- Power Magnitude $= |X(i)|^2$ so (with respect to reference level)

$$
\begin{aligned}
X_{dB} &= 10\log_{10}(|X(i)|^2) \\
&= 20\log_{10}(|X(i)|)
\end{aligned}
$$

which is an expression of dB we often come across.

Decibel and Chillies!

Decibels are used to express wide dynamic ranges in a many applications:



Decibel and acoustics

- dB is commonly used to quantify sound levels relative to some 0 dB reference.
- The reference level is typically set at the *threshold of human perception*
- Human ear is capable of detecting a very large range of sound pressures.

Examples of dB measurement in Sound

Threshold of Pain

The ratio of sound pressure that causes **permanent** damage from short exposure to the limit that (undamaged) ears can hear is above a million:

- The ratio of the maximum power to the minimum power is above one (short scale) trillion (10^{12}).
- The log of a trillion is 12, so this ratio represents a **difference of 120 dB**.
- **120 dB** is the quoted **Threshold of Pain** for Humans.

Examples of dB measurement in Sound (cont.)

Speech Sensitivity

Human ear is not equally sensitive to all the frequencies of sound within the entire spectrum:

- Maximum human sensitivity at noise levels at between 2 and 4 kHz (Speech)
  - These are factored more heavily into sound descriptions using a process called **frequency weighting**.
  - Filter (Partition) into frequency bands concentrated in this range.

  - Used for Speech Analysis
  - Mathematical Modelling of Human Hearing
  - Audio Compression (E.g. **MPEG Audio**)

More on this Later

Examples of dB measurement in Sound (cont.)

Digital Noise increases by 6dB per bit

In digital audio sample representation (**linear pulse-code modulation (PCM)**),

- The first bit (least significant bit, or LSB) produces residual quantization noise (bearing little resemblance to the source signal)
- Each subsequent bit offered by the system **doubles** the resolution, corresponding to a 6 ($= 10 * \log_{10}(4)$) dB.
- So a 16-bit (linear) audio format offers 15 bits beyond the first, for a dynamic range (between quantization noise and clipping) of (15 x 6) = 90 dB, meaning that the maximum signal is 90 dB above the theoretical peak(s) of quantisation noise.
- 8-bit linear PCM similarly gives (7 x 6) = 42 dB.
- 48 dB difference between 8- and 16-bit which is (48/6 (dB)) 8 times as noisy.

More on this Later

Signal to Noise

Signal-to-noise ratio is a term for the power ratio between a signal (meaningful information) and the background noise:

$$SNR = \frac{P_{signal}}{P_{noise}} = \left(\frac{A_{signal}}{A_{noise}}\right)^2$$

where P is average power and A is RMS amplitude.

- Both signal and noise power (or amplitude) must be measured at the same or equivalent points in a system, and within the same system bandwidth.

Because many signals have a very wide dynamic range, SNRs are usually expressed in terms of the logarithmic decibel scale:

$$SNR_{dB} = 10\log_{10}\left(\frac{P_{signal}}{P_{noise}}\right) = 20\log_{10}\left(\frac{A_{signal}}{A_{noise}}\right)$$

System Representation: Algorithms and Signal Flow Graphs

It is common to represent digital system signal processing routines as a visual **signal flow graphs**.

We use a simple *equation* relation to describe the **algorithm**.

Three Basic Building Blocks

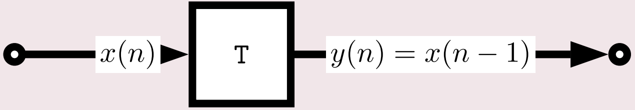
We will need to consider *three* processes:

- Delay
- Multiplication
- Summation

Signal Flow Graphs: Delay

Delay

- We represent a delay of **one sampling interval** by a block with a T **label**:



- We describe the algorithm via the equation: $y(n) = x(n-1)$

Signal Flow Graphs: Delay Example

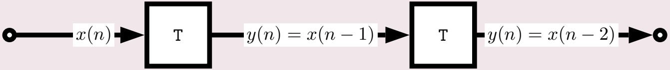
A Delay of 2 Samples

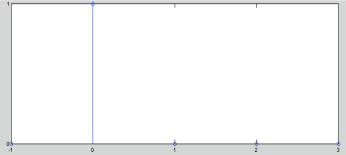
A delay of the input signal by **two** sampling intervals:

- We can describe the **algorithm** by:

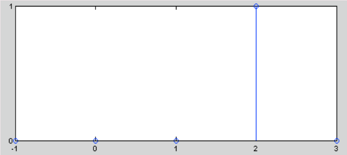
$$\mathbf{y(n) = x(n-2)}$$

- We can use the block diagram to represent the **signal flow graph** as:





$x(n)$

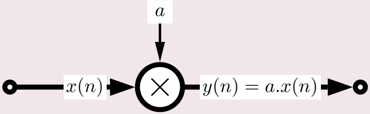


$y(n) = x(n-2)$

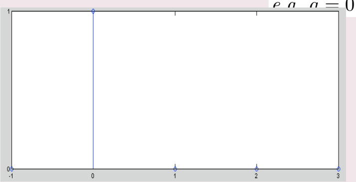
Signal Flow Graphs: Multiplication

Multiplication

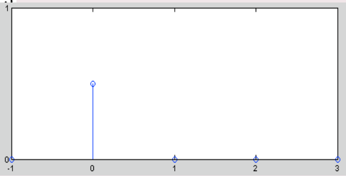
- We represent a multiplication or weighting of the input signal by **a circle with a $\times$ label** .
- We describe the algorithm via the equation: $\mathbf{y(n) = a.x(n)}$



e.g. $a = 0.5$



$x(n)$



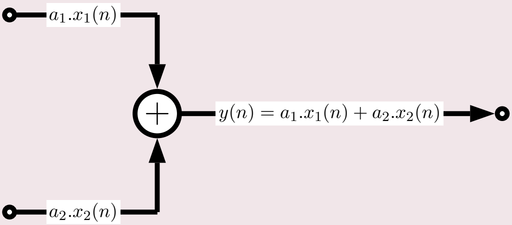
$y(n) = 0.5x(n)$

Signal Flow Graphs: Addition

Addition

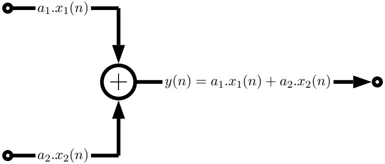
- We represent a addition of two input signal by **a circle with a $+$ label** .
- We describe the algorithm via the equation:

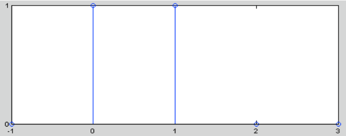
$$\mathbf{y(n) = a_1.x_1(n) + a_2.x_2(n)}$$



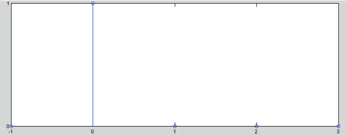
Signal Flow Graphs: Addition Example

In the example, set $a_1 = a_2 = 1$:

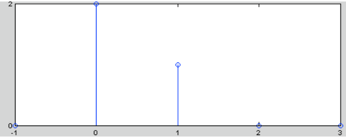




$x_1(n)$



$x_2(n)$



$y(n) = x_1(n) + x_2(n)$

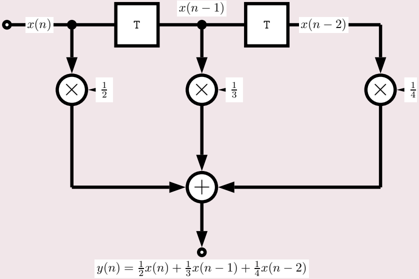
Signal Flow Graphs: Complete Example

All Three Processes Together

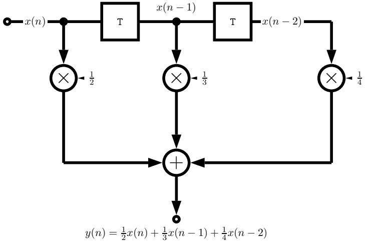
We can combine all above algorithms to build up more complex algorithms:

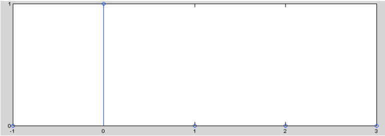
$$y(n) = \frac{1}{2}x(n) + \frac{1}{3}x(n-1) + \frac{1}{4}x(n-2)$$

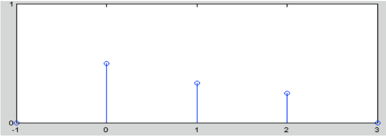
- This has the following signal flow graph:



Signal Flow Graphs: Complete Example Impulse Response







$x(n)$

$y(n) = \frac{1}{2}x(n) + \frac{1}{3}x(n-1) + \frac{1}{4}x(n-2)$

2.2 Filtering

Filtering

Filtering in a broad sense is selecting portion(s) of data for some processing.

If we isolate a portion of data (e.g. audio, image, video) we can

- Remove it — *E.g.* Low Pass, High Pass *etc. filtering*
- Attenuate it — Enhance or diminish its presence, *E.g.* Equalisation, Audio Effects/Synthesis
- Process it in other ways — Digital Audio, *E.g.* Audio Effects/Synthesis

More Later

Filtering Examples (**More Later**)

Filtering Examples:

- In many **multimedia** contexts this involves the removal of data from a signal — This is essential in almost all aspects of **lossy** multimedia data representations.
  - **JPEG Image** Compression
  - **MPEG Video** Compression
  - **MPEG Audio** Compression
- In **Digital Audio** we may wish to determine a range of frequencies we wish the enhance or diminish to equalise the signal, *e.g.*:
  - **Tone** — Treble and Bass — **Controls**
  - **Equalisation (EQ)**
  - **Synthesis** — Subtractive Synthesis, EQ in others.

How can we filter a Digital Signal

Two Ways to Filter

- Temporal Domain — *E.g.* Sampled (PCM) Audio
- Frequency Domain — Analyse frequency components in signal.

We will look at filtering in the **frequency space** very soon, but first we consider filtering in the **temporal domain** via **impulse responses**.

Temporal Domain Filters

We will look at:

IIR Systems : Infinite impulse response systems

FIR Systems : Finite impulse response systems

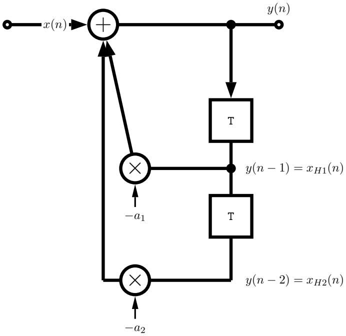
Infinite Impulse Response (IIR) Systems

Simple Example IIR Filter

- The **algorithm** is represented by the **difference equation**:

$$y(n) = x(n) - a_1.y(n-1) - a_2.y(n-2)$$

- This produces the opposite **signal flow graph**

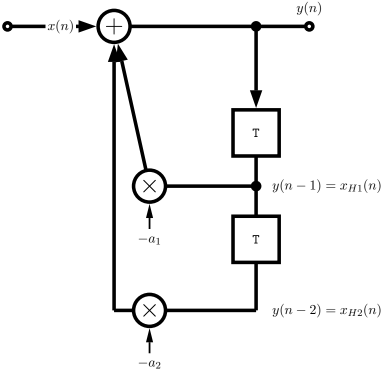


Infinite Impulse Response (IIR)Systems Explained

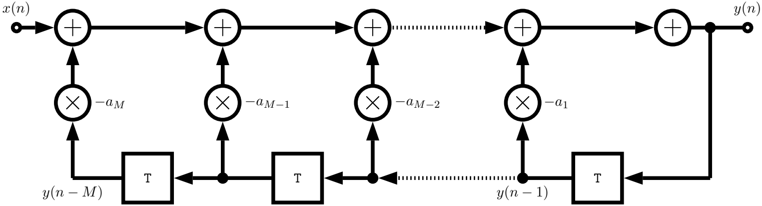
IIR Filter Explained

The following happens:

- The **output signal** $y(n)$ is **fed back** through a **series** of **delays**
- Each **delay** is **weighted**
- Each fed back **weighted delay** is **summed** and passed to **new output**.
- Such a **feedback** system is called a **recursive system**



A Complete IIR System



Complete IIR Algorithm

Here we extend:

The **input** delay line up to $N-1$ elements and

The **output** delay line by M elements.

We can represent the IIR system algorithm by the difference equation:

$$y(n) = x(n) - \sum_{k=1}^{M} a_k y(n-k)$$

Finite Impulse Response (FIR) Systems

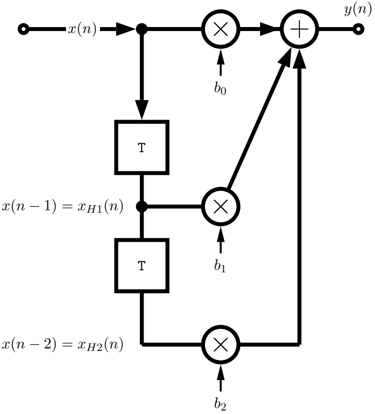
FIR system's are slightly simpler — there is **no feedback loop**.

Simple Example FIR Filter

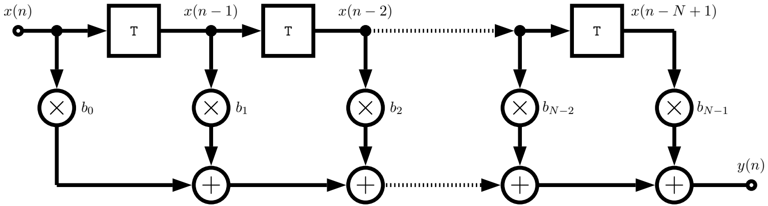
A simple FIR system can be described as follows:

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

- The **input** is **fed through delay elements**
- **Weighted sum** of **delays** gives $y(n)$



A Complete FIR System



FIR Algorithm

To develop a more complete FIR system we need to add $N-1$ **feed forward delays**

We can describe this with the algorithm:

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$$

Filtering with IIR/FIR

We have **two** **filter banks** defined by vectors: $A = \{a_k\}$, $B = \{b_k\}$.

These can be applied in a *sample-by-sample* algorithm:

- MATLAB provides a generic `filter(B,A,X)` function which filters the data in vector `X` with the filter described by vectors `A` and `B` to create the filtered data `Y`.
  The filter is of the standard difference equation form:

$$\begin{aligned} a(1) * y(n) &= b(1) * x(n) + b(2) * x(n-1) + \ldots + b(nb+1) * x(n-nb) \\ &\quad - a(2) * y(n-1) - \ldots - a(na+1) * y(n-na) \end{aligned}$$

- If a(1) is **not equal** to **1**, filter **normalizes** the filter coefficients by a(1). If **a(1) equals 0**, `filter()` **returns** an **error**

Creating Filters

How do I create Filter banks `A` and `B`

- Filter banks can be created manually — Hand Created: **See next slide** and **Equalisation** example later in slides
- MATLAB can provide some predefined filters — **a few slides on, see lab classes**
  - Many standard filters provided by MATLAB
- See also `help filter`, online MATLAB `docs` and lab classes.

Filtering with IIR/FIR: Simple Example

The MATLAB file IIRdemo.m sets up the filter banks as follows:

IIRdemo.m

```
fg=4000;
fa=48000;
k=tan(pi*fg/fa);

b(1)=1/(1+sqrt(2)*k+k^2);
b(2)=-2/(1+sqrt(2)*k+k^2);
b(3)=1/(1+sqrt(2)*k+k^2);
a(1)=1;
a(2)=2*(k^2-1)/(1+sqrt(2)*k+k^2);
a(3)=(1-sqrt(2)*k+k^2)/(1+sqrt(2)*k+k^2);
```

Apply this filter

How to apply the (previous) difference equation:

- By hand

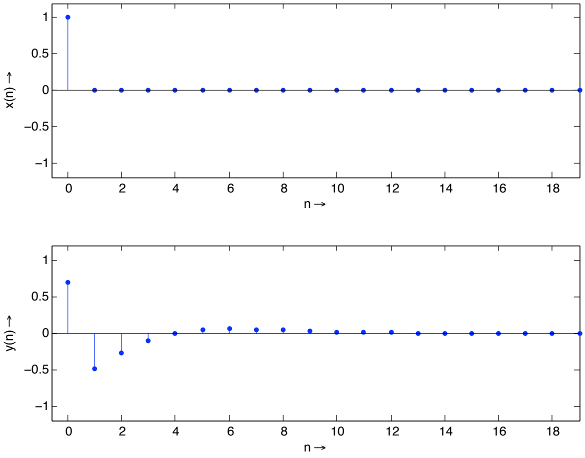
IIRdemo.m cont.

```
for n=1:N
  y(n)=b(1)*x(n) + b(2)*xh1 + b(3)*xh2 ...
          - a(2)*yh1 - a(3)*yh2;
  xh2=xh1;xh1=x(n);
  yh2=yh1;yh1=y(n);
end;
```

- Use MATLAB `filter()` function — **see next but one slide**
  - Far more **preferable**: general — **any length filter**

Filtering with IIR: Simple Example Output

This produces the following output:



MATLAB filters

Matlab `filter()` function implements an IIR/FIR hybrid filter.

Type `help filter`:

```
FILTER One-dimensional digital filter.
    Y = FILTER(B,A,X) filters the data in vector X with the
    filter described by vectors A and B to create the filtered
    data Y.  The filter is a "Direct Form II Transposed"
    implementation of the standard difference equation:

    a(1)*y(n) = b(1)*x(n) + b(2)*x(n-1) + ... + b(nb+1)*x(n-nb)
                          - a(2)*y(n-1) - ... - a(na+1)*y(n-na)

    If a(1) is not equal to 1, FILTER normalizes the filter
    coefficients by a(1).

    FILTER always operates along the first non-singleton dimension,
    namely dimension 1 for column vectors and non-trivial matrices,
    and dimension 2 for row vectors.
```

Using MATLAB to make filters for `filter()` (1)

MATLAB provides a few built-in functions to create ready made filter parameter A and B:

Some common MATLAB Filter Bank Creation Functions

E.g: `butter`, `buttord`, `besself`, `cheby1`, `cheby2`, `ellip`.

See `help` or `doc` appropriate function.

2.3 Fourier Transform (Recap from CM2104/CM2208

The Frequency Domain

The **Frequency domain** can be obtained through the transformation, via **Fourier Transform (FT)**, from

- one **Temporal (Time)** or **Spatial** domain

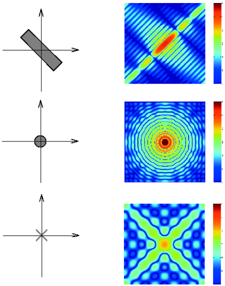
  to the other

- **Frequency** Domain
  - We do not think in terms of signal or pixel intensities but rather underlying sinusoidal waveforms of varying frequency, amplitude and phase.

Applications of Fourier Transform

Numerous Applications including:

- Essential tool for Engineers, Physicists, Mathematicians and Computer Scientists
- Fundamental tool for Digital Signal Processing and Image Processing
- Many types of Frequency Analysis:
  - **Filtering**
  - **Noise Removal**
  - Signal/Image Analysis
  - Simple implementation of **Convolution**
  - **Audio** and Image **Effects Processing**.
  - Signal/Image Restoration — *e.g.* **Deblurring**
  - Signal/Image Compression — **MPEG** (Audio and Video), **JPEG** use related techniques.
  - Many more ......



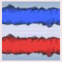
1D Audio Example

Lets consider a 1D (e.g. Audio) example to see what the different domains mean:

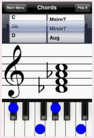
Consider a **complicated sound** such as the a **chord** played on a **piano** or a **guitar**.

We can describe this sound in two related ways:

Temporal Domain : Sample the **amplitude** of the sound many times a second, which gives an approximation to the sound as a **function** of **time**.



Frequency Domain : **Analyse** the sound in terms of the **pitches** of the notes, or **frequencies**, which make the sound up, recording the **amplitude** of **each frequency**.



Fundamental Frequencies

D♭	: 554.40Hz
F	: 698.48Hz
A♭	: 830.64Hz
C:	1046.56Hz

plus harmonics/partial frequencies ....

Back to Basics

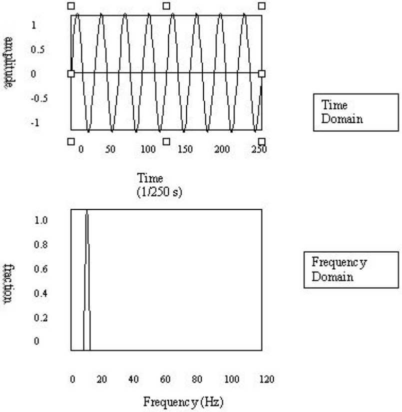
An 8 Hz Sine Wave

A signal that consists of a **sinusoidal** wave at **8 Hz**.

- 8 Hz means that wave is completing 8 cycles in 1 second
- The **frequency** of that wave is 8 Hz.

From the **frequency domain** we can see that the composition of our signal is

- **one peak** occurring with a frequency of 8 Hz — there is only one sine wave here.
  - with a **magnitude/fraction** of **1.0** i.e. it is the **whole signal**.



2D Image Example

What do Frequencies in an Image Mean?

Now images are no more complex really:

- **Brightness** along a **line** can be recorded as a set of **values** measured at **equally** spaced **distances apart**,
- **Or** equivalently, at a **set** of **spatial frequency values**.
- Each of these frequency values is a **frequency component**.
- An image is a 2D array of pixel measurements.
- We form a 2D grid of spatial frequencies.
  - A given frequency component now specifies what contribution is made by data which is changing with specified x and y direction spatial frequencies.

Frequency components of an image

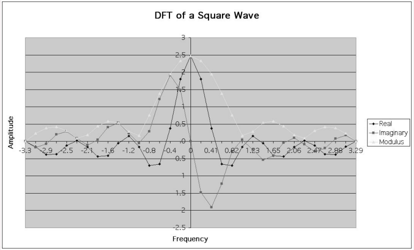
What do Frequencies in an Image Mean? (Cont.)

- Large values at **high** frequency components then the data is changing rapidly on a short distance scale.
  - *e.g.* a **page of text**
  - **However**, **Noise** contributes (very) **High Frequencies** also
- Large **low** frequency components then the large scale features of the picture are more important.
  e.g. a single fairly simple object which occupies most of the image.

Visualising Frequency Domain Transforms

Sinusoidal Decomposition

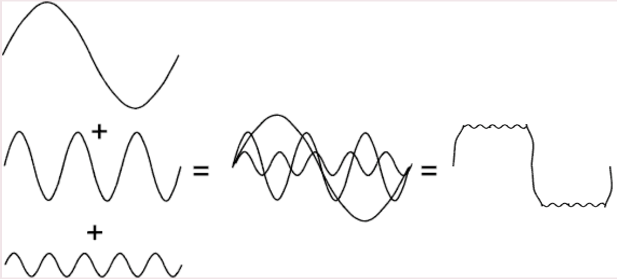
- **Any digital signal** (function) can be **decomposed** into purely **sinusoidal components**
  - Sine waves of different size/shape — varying **amplitude**, **frequency** and **phase**.
- When **added** back **together** they **reconstitute** the **original signal**.
- The **Fourier transform** is the tool that performs such an operation.



Summing Sine Waves. Example: to give a Square(ish) Wave (**E.g. Additive Synthesis**)

Digital signals are composite signals made up of many sinusoidal frequencies

- A 200Hz digital signal (**square(ish) wave**) may be a composed of 200, 600, 1000, *etc.* sinusoidal signals which sum to give:



Summary so far

So What Does All This Mean?

Transforming a signal into the frequency domain allows us

- **To see what sine waves make up our underlying signal**
- **E.g.**
  - One part sinusoidal wave at 50 Hz and
  - Second part sinusoidal wave at 200 Hz.
  - *Etc.*
- More **complex** signals will give more complex decompositions but the idea is exactly the same.

How is this Useful then?

Basic Idea of Filtering in Frequency Space

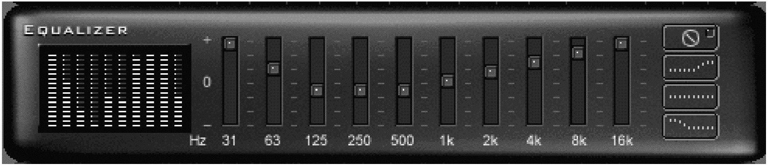
Filtering now involves **attenuating** or **removing** certain frequencies — **easily performed**:

- **Low pass filter** —
  - **Ignore high frequency** noise components — make **zero** or a **very low value**.
  - Only store lower frequency components
- **High Pass Filter** — **opposite of above**
- **Bandpass Filter** — only **allow** frequencies in a **certain range**.

Visualising the Frequency Domain

Think Graphic Equaliser

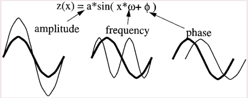
An easy way to visualise what is happening is to think of a graphic equaliser on a stereo system (or some software audio players, *e.g. iTunes*).



So are we ready for the Fourier Transform?

We have all the Tools....

- This lecture, so far, (hopefully) set the context for Frequency decomposition.
- Past Maths Lectures:
  - **Odd/Even Functions**: $\sin(-x) = -\sin(x)$, $\cos(-x) = \cos(x)$
  - **Complex Numbers**: **Phasor Form** $re^{i\phi} = r(\cos\phi + i\sin\phi)$
  - Calculus **Integration**: $\int e^{kx}dx = \frac{e^{kx}}{k}$
- Digital Signal Processing:
  - Basic Waveform Theory. Sine Wave $y = A.sin(2\pi.n.F_w/F_s)$ where: $A =$ amplitude, $F_w =$ wave frequency, $F_s =$ sample frequency, n is the sample index.
  - Relationship between Amplitude, Frequency and Phase:



    - Cosine is a Sine wave 90° out of phase
  - Impulse Responses
- DSP + Image Proc.: Filters and other processing, Convolution

Fourier Theory

Introducing The Fourier Transform

The tool which **converts** a **spatial** or **temporal** (real space) **description** of **audio**/**image** data, for example, into one in terms of its **frequency components** is called the **Fourier transform**

The new version is usually referred to as the **Fourier space description** of the data.

We then essentially process the data:

- *E.g.* for **filtering** basically this means attenuating or setting certain frequencies to zero

We then need to **convert data back** (or **invert**) to **real audio**/imagery to use in our applications.

The corresponding **inverse** transformation which turns a Fourier space description back into a real space one is called the **inverse Fourier transform**.

1D Fourier Transform

1D Case (*e.g.* Audio Signal)

Considering a **continuous** function $f(x)$ of a single variable x representing distance (or time).

The **Fourier transform** of that function is denoted $F(u)$, where u represents **spatial** (or **temporal**) **frequency** is defined by:

$$F(u) = \int_{-\infty}^{\infty} f(x)\mathbf{e}^{-\mathbf{2\pi ixu}}\, dx.$$

Note: In general $F(u)$ will be a complex quantity *even though* the original data is purely **real**.

- The meaning of this is that not only is the **magnitude** of each **frequency** present important, but that its **phase relationship** is **too**.
- Recall **Phasors** from **Complex Number Lectures**.
  - $e^{-2\pi ixu}$ above is a **Phasor**.

Inverse Fourier Transform

Inverse 1D Fourier Transform

The **inverse Fourier transform** for regenerating $f(x)$ from $F(u)$ is given by

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{\mathbf{2\pi ixu}}\, du,$$

which is rather similar to the (forward) Fourier transform

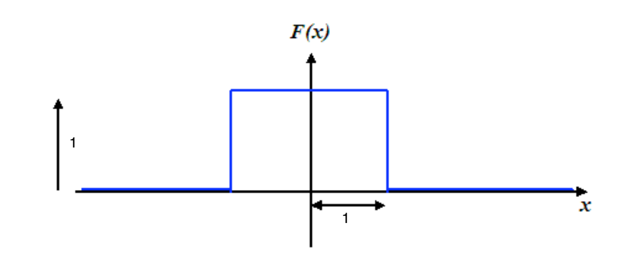
- except that the **exponential term has the opposite sign.**
- It is **not negative**

Fourier Transform Example

Fourier Transform of a Top Hat Function

Let's see how we compute a Fourier Transform: consider a particular function $f(x)$ defined as

$$f(x) = \begin{cases} 1 & \textbf{if } |x| \leq 1 \\ 0 & \textbf{otherwise,} \end{cases}$$



The Sinc Function (1)

We derive the Sinc function

So its Fourier transform is:

$$\begin{aligned} \mathbf{F(u)} &= \int_{-\infty}^{\infty} f(x) e^{-2\pi ixu}\, dx \\ &= \int_{-1}^{1} 1 \times e^{-2\pi ixu}\, dx \\ &= \frac{-1}{2\pi iu}(e^{2\pi iu} - e^{-2\pi iu}) \end{aligned}$$

Now (refer to **Complex Numbers** Lectures/**Maths Formula Sheet** Handout)

$$\begin{aligned} \sin\theta &= \frac{\mathbf{e^{i\theta} - e^{-i\theta}}}{\mathbf{2i}}, \text{So:} \\ \mathbf{F(u)} &= \frac{\sin \mathbf{2\pi u}}{\mathbf{\pi u}}. \end{aligned}$$

In this case, $F(u)$ is purely real, which is a consequence of the original data being **symmetric** in x and $-x$.

- $f(x)$ is an **even** function.

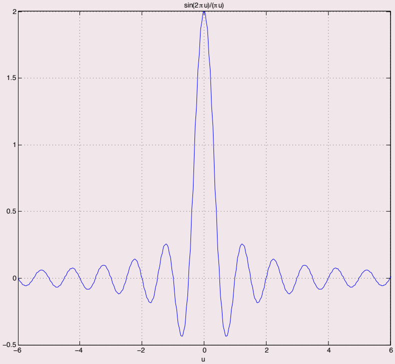
A graph of $F(u)$ is shown overleaf.

This function is often referred to as the **Sinc function**.

The Sinc Function Graph

The Sinc Function

The Fourier transform of a top hat function, the **Sinc function**:



The 2D Fourier Transform

2D Case (*e.g.* Image data)

If $f(x, y)$ is a function, for example **intensities** in an **image**, its **Fourier transform** is given by

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\mathbf{e}^{-\mathbf{2\pi i(xu+yv)}}\, dx\, dy,$$

and the **inverse transform**, as might be expected, is

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v)e^{\mathbf{2\pi i(xu+yv)}}\, du\, dv.$$

The Discrete Fourier Transform

But All Our Audio and Image data are Digitised!!

Thus, we need a *discrete* formulation of the Fourier transform:

- **Assumes** **regularly spaced** data values, and
- **Returns** the **value** of the Fourier transform for a set of values in frequency space which are **equally spaced**.

This is done quite naturally by replacing the integral by a summation, to give the *discrete Fourier transform* or **DFT** for short.

1D Discrete Fourier transform

1D Case:

In 1D it is convenient now to assume that x goes up in steps of 1, and that there are N samples, at values of x from 0 to $N-1$.

So the DFT takes the form

$$F(u) = \frac{\mathbf{1}}{\mathbf{N}} \sum_{x=0}^{N-1} f(x) e^{-2\pi ixu/\mathbf{N}},$$

while the inverse DFT is

$$f(x) = \sum_{x=0}^{N-1} F(u) e^{2\pi ixu/\mathbf{N}}.$$

NOTE: Minor changes from the continuous case are a factor of $1/N$ in the **exponential** terms, and also the factor $1/N$ in front of the forward transform which **does not appear** in the **inverse** transform.

2D Discrete Fourier transform

2D Case

The **2D DFT** works is similar.

So for an $N \times M$ grid in x and y we have

$$F(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{1}}{\mathbf{NM}} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) e^{-2\pi i (x\mathbf{u}/\mathbf{N} + y\mathbf{v}/\mathbf{M})},$$

and

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u, v) e^{2\pi i (xu/\mathbf{N} + yv/\mathbf{M})}.$$

Balancing the 2D DFT

Most Images are Square

Often $N = M$, and it is then it is more convenient to redefine $F(u, v)$ by multiplying it by a factor of N, so that the **forward** and **inverse** transforms are more **symmetric**:

$$F(u, v) = \frac{\mathbf{1}}{\mathbf{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) e^{-2\pi i(xu+yv)/\mathbf{N}},$$

and

$$f(x, y) = \frac{\mathbf{1}}{\mathbf{N}} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u, v) e^{2\pi i(xu+yv)/\mathbf{N}}.$$

Fourier Transforms in MATLAB

`fft()` and `fft2()`

MATLAB provides functions for 1D and 2D **Discrete Fourier Transforms** (**DFT**):

fft(X)
: is the 1D discrete Fourier transform (DFT) of **vector** X. For **matrices**, the FFT operation is applied to **each column** — **NOT** a 2D DFT transform.

fft2(X)
: returns the 2D Fourier transform of matrix X. If X is a vector, the result will have the same orientation.

fftn(X)
: returns the N-D discrete Fourier transform of the **N-D array X**.

Inverse DFT
: **ifft()**, **ifft2()**, **ifftn()** perform the **inverse** DFT.

See appropriate MATLAB **help/doc** pages for **full details**.

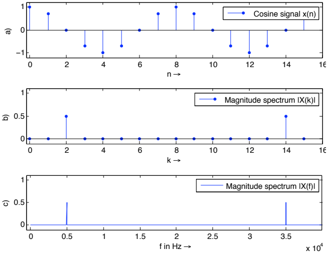
Plenty of examples to Follow.

Visualising the Fourier Transform

Visualising the Fourier Transform

Having computed a DFT it might be useful to visualise its result:

- It's useful to visualise the Fourier Transform
- Standard tools
- Easily plotted in MATLAB



The Magnitude Spectrum of Fourier Transform

Recall that the Fourier Transform of our **real** audio/image data is always **complex**

- **Phasors**: This is how we encode the **phase** of the underlying signal's **Fourier Components**.

How can we visualise a complex data array?

Back to Complex Numbers:

Magnitude spectrum **Compute the absolute value of the complex data**:

$$|F(k)| = \sqrt{F_R^2(k) + F_I^2(k)} \textbf{ for } k = 0, 1, \ldots, N-1$$

where $F_R(k)$ is the **real** part and $F_I(k)$ is the **imaginary** part of the N sampled Fourier Transform, $F(k)$.

Recall MATLAB: `Sp = abs(fft(X,N))/N;` (**Normalised form**)

The Phase Spectrum of Fourier Transform

The Phase Spectrum

Phase Spectrum

The Fourier Transform also represent phase, the **phase spectrum** is given by:

$$\varphi = \arctan \frac{F_I(k)}{F_R(k)} \textbf{ for } k = 0, 1, \ldots, N-1$$

Recall MATLAB: `phi = angle(fft(X,N))`

Relating a Sample Point to a Frequency Point

When **plotting graphs** of *Fourier Spectra* and doing other DFT processing we may wish to **plot** the x-axis in **Hz** (**Frequency**) rather than **sample point** number $k = 0, 1, \ldots, N-1$

There is a **simple relation** between the two:

- The sample points go in steps $k = 0, 1, \ldots, N-1$
- For a given sample point k the frequency relating to this is given by:

$$f_k = k\frac{f_s}{N}$$

  where f_s is the *sampling frequency* and N the **number** of samples.
- Thus we have **equidistant frequency steps** of $\frac{f_s}{N}$ ranging from 0 Hz to $\frac{N-1}{N}f_s$ Hz

Time-Frequency Representation: Spectrogram

Spectrogram

It is often **useful** to look at the **frequency distribution** over a **short-time**:

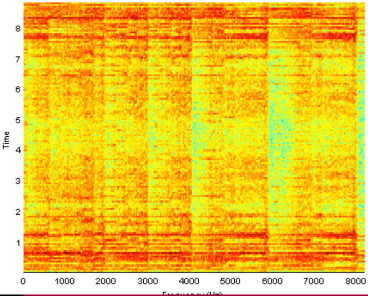
- Split signal into N segments
- Do a **windowed Fourier Transform** — **Short-Time Fourier Transform** (**STFT**)
  - Window needed to reduce *leakage* effect of doing a shorter sample SFFT.
  - Apply a **Blackman**, **Hamming** or **Hanning** Window
- MATLAB function does the job: `Spectrogram` — see `help spectrogram`
- See also MATLAB's `specgramdemo`

MATLAB spectrogram Example

spectrogrameg.m

```
load('handel')
[N M] = size(y);
figure(1)
spectrogram(y,512,20,1024,Fs);
```

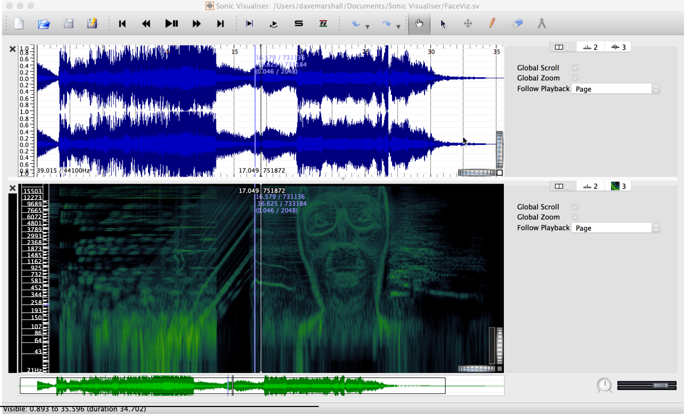
Produces the following:



Aphex Twin Spectrogram

Aphex Twin famously[1] embedded images in the spectrogram of a few tracks on his Windowlicker EP. His face on Track 2 "Formula" or "Equation" (Full title:

$\Delta M_{i-1} = -\alpha \sum_{n=1}^{N} D_i[n][\sum_{\sigma \in C[i]} F_{ji}[n-1] + Fext_i[n-1]]$!!:



[1] See here for web link to other examples of embedded image

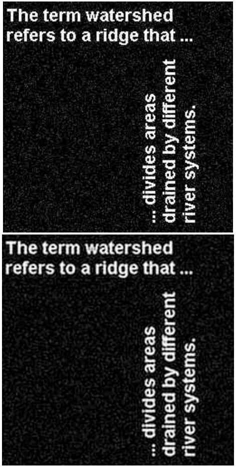
Filtering in the Frequency Domain

Low Pass Filter

Example: *Audio Hiss, 'Salt and Pepper' noise in images,*

Noise:

- The idea with **noise Filtering** is to reduce various spurious effects of a **local nature** in the image, caused perhaps by
  - **noise** in the acquisition system,
  - arising as a result of **transmission** of the data, for example from a space probe utilising a low-power transmitter.



Frequency Space Filtering Methods

Low Pass Filtering — Remove Noise

Noise = High Frequencies:

- In audio data many spurious peaks in over a short timescale.
- In an image means there are many rapid transitions (over a short distance) in intensity from high to low and back again or vice versa, as faulty pixels are encountered.
- **Not all high frequency data noise though!**

Therefore **noise** will contribute heavily to the **high frequency** components of the signal when it is **analysed** in **Fourier space**.

Thus if we **reduce** the **high frequency** components — **Low-Pass Filter** should (if tuned properly) **reduce** the amount of noise in the data.

(Low-pass) Filtering in the Fourier Space

Low Pass Filtering with the Fourier Transform

We **filter** in Fourier space by computing

$$G(u, v) = H(u, v)F(u, v)$$

where:

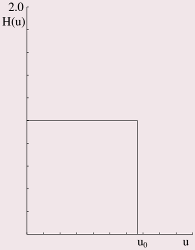
- $F(u, v)$ is the **Fourier transform** of the **original** image,
- $H(u, v)$ is a filter function, designed to reduce high frequencies, and
- $G(u, v)$ is the **Fourier transform of the improved image**.
- **Inverse Fourier transform** $G(u, v)$ to get $g(x, y)$ our **improved image**

Ideal Low-Pass Filter

We need to design or compute $H(u, v)$

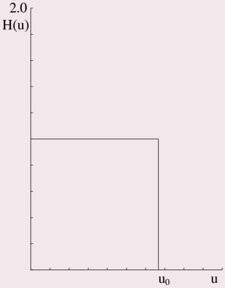
- If we know $h(x, y)$ or have a discrete sample of $h(x, y)$ can compute its Fourier Transform
- Can simply design simple filters in Frequency Space

The simplest sort of filter to use is an *ideal low-pass filter*, which in one dimension appears as :



Ideal Low-Pass Filter (2)

How the Low Pass Filter Works with Frequencies



This is a function, $H(u)$, which is a top-hat **1** for u between 0 and u_0, the *cut-off frequency*, and **zero** elsewhere.

- So all frequency space information **above** u_0 is **discarded**, and all information **below** u_0 is **kept**.
- A **very simple** computational process.

Ideal 2D Low-Pass Filter

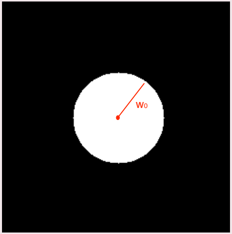
Ideal 2D Low-Pass Filter

The two dimensional version of this is the Low-Pass Filter:

$$H(u,v) = \begin{cases} 1 & \textbf{if } \sqrt{u^2+v^2} \leq w_0 \\ 0 & \textbf{otherwise,} \end{cases}$$

where w_0 is now the **cut-off frequency** for **both** dimensions.

- Thus, **all** frequencies **inside** a **radius** w_0 are **kept**, and **all** others **discarded**.



Not So Ideal Low-Pass Filter? (1)

In practice, the ideal Low-Pass Filter is no so ideal

The **problem** with this filter is that as well as noise there may be **useful** high frequency content:

- In **audio**: plenty of other high frequency content: high pitches, rustles, scrapes, wind, mechanical noises, cymbal crashes etc.
- In **images**: **edges** (places of rapid transition from light to dark) also significantly contribute to the high frequency components.

Choosing the **most appropriate** cut-off frequency is not so easy

- Similar problem to choosing a threshold in **image thresholding**.

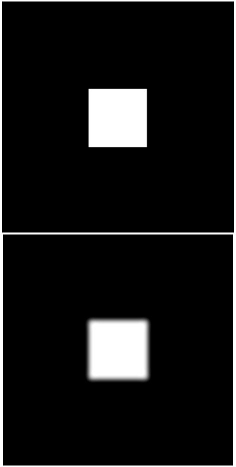
Not So Ideal Low-Pass Filter? (2)

What if you set the wrong value for the cut-off frequency?

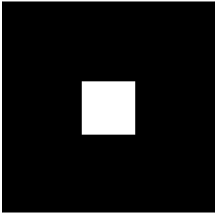
If you **choose the wrong cut-off frequency** an ideal low-pass filter will tend to *blur* the data:

- High audio frequencies become muffled
- Edges in images become blurred.

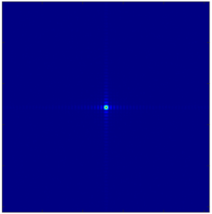
The lower the cut-off frequency is made, the more pronounced this effect becomes in *useful data content*



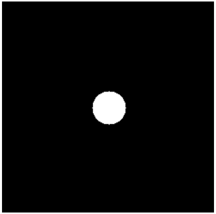
Ideal Low Pass Filter Example 1



(a) Input Image



(b) Image Spectra



(c) Ideal Low Pass Filter

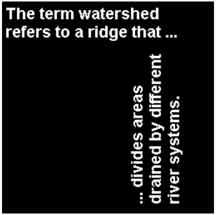
(d) Filtered Image

Ideal Low-Pass Filter Example 1 MATLAB Code

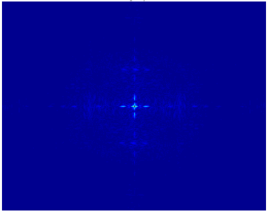
lowpass.m:

```
% Create a white box on a
% black background image
M = 256; N = 256;
image = zeros(M,N)
box = ones(64,64);
%box at centre
image(97:160,97:160) = box;

% Show Image

figure(1);
imshow(image);

% compute fft and display its spectra

F=fft2(double(image));
figure(2);
imagesc((abs(fftshift(F))/(M*N)));
colormap(jet);
axis off;

% Compute Ideal Low Pass Filter
u0 = 20; % set cut off frequency

u=0:(M-1);
v=0:(N-1);
idx=find(u>M/2);
u(idx)=u(idx)-M;
idy=find(v>N/2);
v(idy)=v(idy)-N;
[V,U]=meshgrid(v,u);
D=sqrt(U.^2+V.^2);
H=double(D<=u0);

% display
figure(3);
imshow(fftshift(H));

% Apply filter and do inverse FFT
G=H.*F;
g=real(ifft2(double(G)));

% Show Result
figure(4);
imshow(g);
```

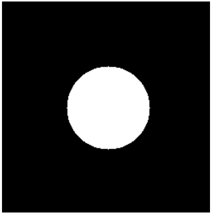
Ideal Low Pass Filter Example 2



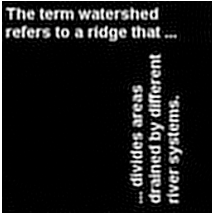
(a) Input Image



(b) Image Spectra



(c) Ideal Low-Pass Filter



(d) Filtered Image

Ideal Low-Pass Filter Example 2 MATLAB Code

lowpass2.m:

```
% read in MATLAB demo text image
image = imread('text.png');
[M N] = size(image)


% Show Image

figure(1);
imshow(image);

% compute fft and display its spectra

F=fft2(double(image));
figure(2);
imagesc((abs(fftshift(F))/(M*N)));
colormap(jet);
axis off;
```

```
% Compute Ideal Low Pass Filter
u0 = 50; % set cut off frequency

u=0:(M-1);
v=0:(N-1);
idx=find(u>M/2);
u(idx)=u(idx)-M;
idy=find(v>N/2);
v(idy)=v(idy)-N;
[V,U]=meshgrid(v,u);
D=sqrt(U.^2+V.^2);
H=double(D<=u0);

% display
figure(3);
imshow(fftshift(H));

% Apply filter and do inverse FFT
G=H.*F;
g=real(ifft2(double(G)));

% Show Result
figure(4);
imshow(g);
```

Low-Pass Butterworth Filter (1)

We introduced the **Butterworth Filter** with **IIR/FIR Filters** (**Temporal Domain Filtering**). Let's now study it in more detail.

- Much easier to visualise in Frequency space

2D Low-Pass Butterworth Filter

Another popular (and general) filter is the **Butterworth low pass filter**.

In the 2D case, $H(u, v)$ takes the form

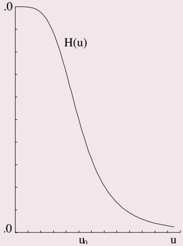
$$H(u, v) = \frac{1}{1 + [(u^2 + v^2)/w_0^2]^n},$$

where n is called the **order** of the filter.

Low-Pass Butterworth Filter (2)

Visualising the 1D Low-Pass Butterworth Filter

This keeps some of the high frequency information, as illustrated by the second order **one dimensional** Butterworth filter:



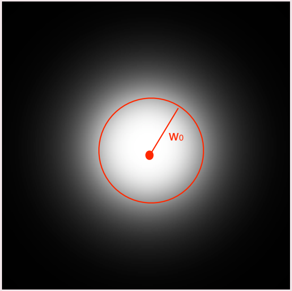
Consequently **reduces** the **blurring**.

- **Blurring** the **filter** — Butterworth is essentially a **smoothed** top hat functions — **reduces blurring by** the filter.

Low-Pass Butterworth Filter (3)

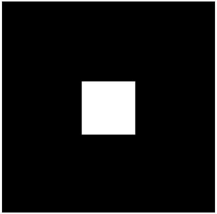
Visualising the 2D Low-Pass Butterworth Filter

The **2D second order** Butterworth filter looks like this:

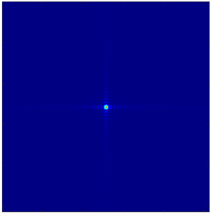


- Note this is **blurred circle** — blurring of the ideal 2D Low-Pass Filter.

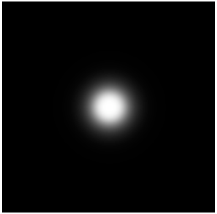
Butterworth Low Pass Filter Example 1 (1)



(a) Input Image



(b) Image Spectra



(c) Butterworth Low-Pass Filter

(d) Filtered Image

Butterworth Low-Pass Filter Example 1 (2)

Comparison of Ideal and Butterworth Low Pass Filter:

Ideal Low-Pass

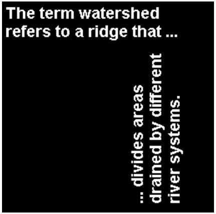
Butterworth Low-Pass

Butterworth Low-Pass Filter Example 1 (3)

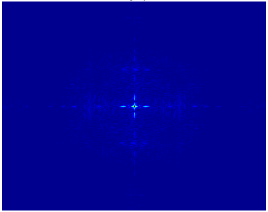
butterworth.m:

```
% Load Image and Compute FFT as
%  in Ideal Low Pass Filter Example 1
.......
% Compute Butterworth Low Pass Filter
u0 = 20; % set cut off frequency

u=0:(M-1);
v=0:(N-1);
idx=find(u>M/2);
u(idx)=u(idx)-M;
idy=find(v>N/2);
v(idy)=v(idy)-N;
[V,U]=meshgrid(v,u);

for i = 1: M
    for j = 1:N
      %Apply a 2nd order Butterworth
      UVw = double((U(i,j)*U(i,j) + V(i,j)*V(i,j))/(u0*u0));
      H(i,j) = 1/(1 + UVw*UVw);
    end
end
% Display Filter and Filtered Image as before
```

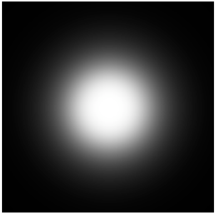
Low-Pass Butterworth Filter Example 2 (1)



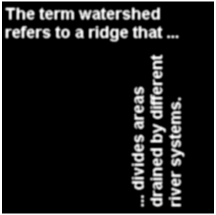
(a) Input Image



(b) Image Spectra



(c) Butterworth Low-Pass Filter

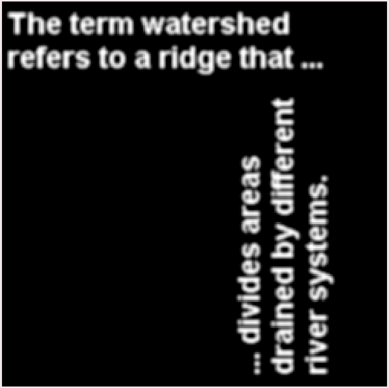


(d) Filtered Image

Low-Pass Butterworth Filter Example 2 (2)

Comparison of Ideal and Low-Pass Butterworth Filter:

Ideal Low Pass



Butterworth Low-Pass

Butterworth Low Pass Filter Example 2 MATLAB (3)

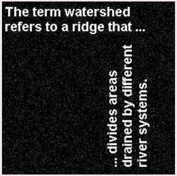
butterworth2.m:

```
% Load Image and Compute FFT as in Ideal Low Pass Filter
% Example 2
.......
% Compute Butterworth Low Pass Filter
u0 = 50; % set cut off frequency

u=0:(M-1);
v=0:(N-1);
idx=find(u>M/2);
u(idx)=u(idx)-M;
idy=find(v>N/2);
v(idy)=v(idy)-N;
[V,U]=meshgrid(v,u);

for i = 1: M
    for j = 1:N
      %Apply a 2nd order Butterworth
      UVw = double((U(i,j)*U(i,j) + V(i,j)*V(i,j))/(u0*u0));
      H(i,j) = 1/(1 + UVw*UVw);
    end
end
% Display Filter and Filtered Image as before
```

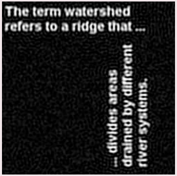
Low Pass Filtering Noisy Images

How to create noise and results of Low Pass Filtering

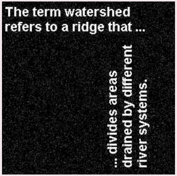
Use Matlab function, `imnoise()` to add noise to image (lowpass.m, lowpass2.m):



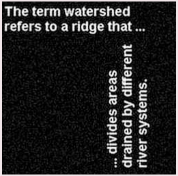
(a) Input Noisy Image



(b) Deconvolved Noisy Image (Low Cut Off)



(c) Input Noisy Image



(d) Deconvolved Noisy Image (Higher Cut Off)

Other Filters

Other Filters

High-Pass Filters — opposite of low-pass, select high frequencies, attenuate those **below** u_0

Band-pass — allow frequencies in a range $u_0 \dots u_1$ attenuate those outside this range

Band-reject — opposite of band-pass, attenuate frequencies within $u_0 \dots u_1$ **select** those **outside** this range

Notch — attenuate frequencies in a narrow bandwidth around cut-off frequency, u_0

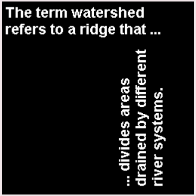
Resonator — amplify frequencies in a narrow bandwidth around cut-off frequency, u_0

Other filters exist that essentially are a combination/variation of the above

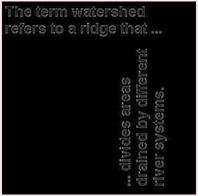
High Pass Filtering

Easy to Implement from the above Low Pass Filter

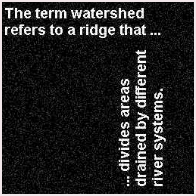
A High Pass Filter is usually defined as **1 - low pass** $= 1 - H$:



(a) Input Image



(b) High Pass Filtered Image



(c) Input Noisy Image

(d) High Pass Filtered Noisy Image

Convolution

Many Useful Applications of Convolution

Several important audio and optical effects can be described in terms of convolutions.

- FIltering — In fact the **above Fourier filtering** is applying **convolutions** of a **low pass filter** where the equations are Fourier Transforms of real space equivalents.
- Deblurring — **high pass** filtering
- Reverb — impulse response convolution (**more soon**).

Note we have seen a discrete **real domain** example of Convolution with **Edge Detection**.

Formal Definition of 1D Convolution:

Let us examine the concepts using 1D continuous functions.

The convolution of two functions $f(x)$ and $g(x)$, written $f(x) * g(x)$, is defined by the integral

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)\,d\alpha.$$

- $*$ is the mathematical **notation** for **convolution**.

No Fourier Transform in sight here — **but wait**!

1D Convolution Real Domain Example (1)

Convolution of Two Top Hat Functions

For example, let us take two **top hat functions**:

Let $f(\alpha)$ be the top hat function shown:

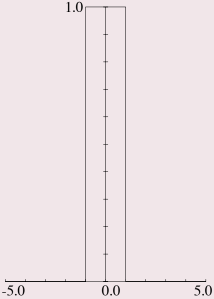
$$f(\alpha) = \begin{cases} 1 & \textbf{if } |\alpha| \leq 1 \\ 0 & \textbf{otherwise,} \end{cases}$$

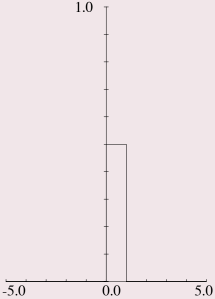
and let $g(\alpha)$ be as shown in next slide, defined by

$$g(\alpha) = \begin{cases} 1/2 & \textbf{if } 0 \leq \alpha \leq 1 \\ 0 & \textbf{otherwise.} \end{cases}$$

1D Convolution Example (2)

Our Two Top Hat Functions Plots



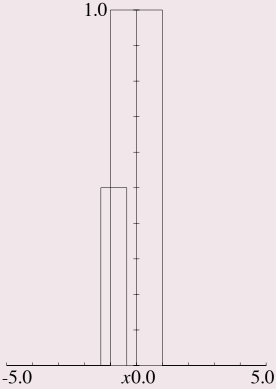


$$f(\alpha) = \begin{cases} 1 & \textbf{if } |\alpha| \leq 1 \\ 0 & \textbf{otherwise,} \end{cases} \qquad g(\alpha) = \begin{cases} 1/2 & \textbf{if } 0 \leq \alpha \leq 1 \\ 0 & \textbf{otherwise.} \end{cases}$$

1D Convolution Example (3)

The Convolution Process: Graphical Interpretation

- $g(-\alpha)$ is the **reflection** of this function in the **vertical** y-axis,
- $g(x-\alpha)$ is the **latter shifted** to the right by a **distance** x.
- Thus for a given value of x, $f(\alpha)g(x-\alpha)$ integrated over all α is the area of overlap of these two top hats, as $f(\alpha)$ has unit height.
- An example is shown for x in the range $-1 \leq x \leq 0$ opposite



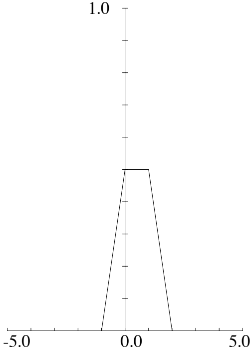
1D Convolution Example (4)

So the solution is:

If we now consider x moving from $-\infty$ to $+\infty$, we can see that

- For $x \leq -1$ or $x \geq 2$, there is **no overlap**;
- As x goes from -1 to 0 the area of overlap **steadily increases** from 0 to $1/2$;
- As x **increases** from0 to 1, the overlap area remains at $1/2$;
- Finally as x increases from 1 to 2, the overlap area steadily **decreases** again from $1/2$ to 0.
- Thus the convolution of $f(x)$ and $g(x)$, $f(x) * g(x)$, in this case has the form shown on next slide

1D Convolution Example (5)

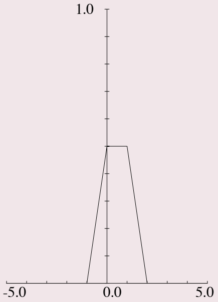


Result of $f(x) * g(x)$

1D Convolution Example (6)

Mathematically the convolution is expressed by:

$$f(x) * g(x) = \begin{cases} (x+1)/2 & \textbf{if } -1 \leq x \leq 0 \\ 1/2 & \textbf{if } 0 \leq x \leq 1 \\ 1 - x/2 & \textbf{if } 1 \leq x \leq 2 \\ 0 & \textbf{otherwise.} \end{cases}$$



Fourier Transforms and Convolution

Convolution Theorem: Convolution in Frequency Space is Easy

One **major** reason that Fourier transforms are so important in signal/image processing is the **convolution theorem** which states that:

If $f(x)$ and $g(x)$ are two functions with Fourier transforms $F(u)$ and $G(u)$, then the Fourier transform of the convolution $f(x) * g(x)$ is simply the product of the Fourier transforms of the two functions, $F(u)G(u)$.

Fourier Transforms and Convolution (Cont.)

Recall our Low Pass Filter Example (MATLAB CODE)

```
% Apply filter
G=H.*F;
```

Where `F` was the Fourier transform of the image, `H` the filter

Computing Convolutions with the Fourier Transform

Example Applications:

- To apply some reverb to an audio signal.
- To compensate for a less than ideal image capture system.

More soon.

Example Applications (Cont.)

Deconvolution: Compensating for undesirable effects

To do this **fast convolution** we simply:

- Take the **Fourier transform** of the **audio/imperfect image**,
- Take the **Fourier transform** of the **function describing the effect** of the system,
- **Multiply** by the effect to apply effect to audio data
- To **remove**/**compensate** for effect: Divide by the effect to obtain the Fourier transform of the ideal image.
- **Inverse** Fourier transform to **recover** the new **improved** audio image.

This process is sometimes referred to as **deconvolution**.

Image Deblurring Deconvolution Example

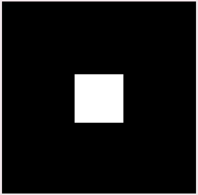
Inverting our Previous Low-Pass Filter

Recall our Low Pass (Butterworth) Filter example of a few slides ago: butterworth.m:
deconv.m and deconv2.m reuses this code and adds a deconvolution stage:

- Our computed butterworth low pass filter, H is our blurring function
- So to simply invert this we can divide (as opposed to multiply) by H with the blurred image G — effectively a **high pass filter**

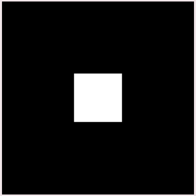
```
Ghigh = G./H;
ghigh=real(ifft2(double(Ghigh)));
figure(5)
imshow(ghigh)
```

- In this ideal example we clearly get F back and to get the image simply to inverse Fourier Transfer.
- In the real world we don't really know the **exact blurring function** H so things are not so easy.



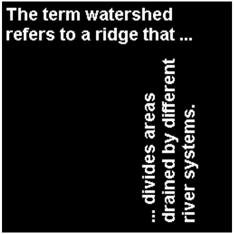
(a) Input Image

(b) Blurred Low-Pass Filtered Image

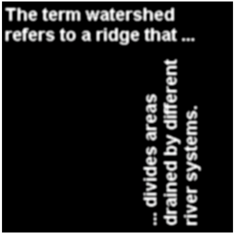


(c) Deconvolved Image

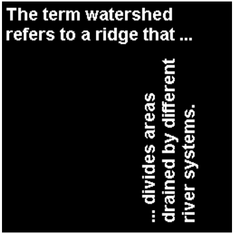
deconv2.m results



(a) Input Image

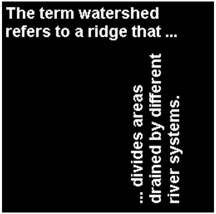


(b) Blurred Low-Pass Filtered Image

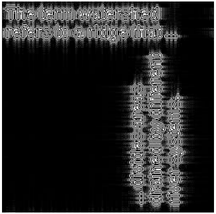


(c) Deconvolved Image

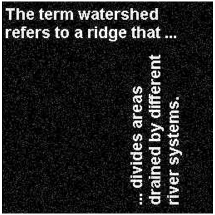
Deconvolution is not always that simple!



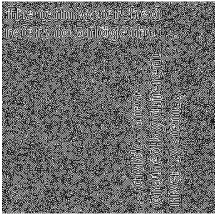
(a) Input Image



(b) Deconvolved Image



(c) Input Noisy Image



(d) Deconvolved Noisy Image