Classification/Comparison of Curves by an Infinite Family of Shape Invariants

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***Abstract*—In this paper we start with a family of boundary based shape measures $I_N(\gamma) = \int_\gamma (x(s)^2 + y(s)^2)^N ds$, $N = 1, 2, \ldots$, defined for every curve γ given in an arc-length parametrisation $x = x(s)$, $y = y(s)$, $s \in [0, 1]$ and placed such that the centroid of γ and the origin coincide. We prove**

$$I_N(\gamma) \le 4^{-N}, \quad \textbf{for all} \quad N = 1, 2, \ldots$$

which implies that the sequence $I_N(\gamma)$ converges quickly to 0 and, therefore the first few measures $I_N(\gamma)$ are most useful to compare shapes and to be applied in tasks like object classification, recognition or identification.

In order to overcome such a problem, we modify the family $I_N(\gamma)$ and also introduce a parameter p to define a new family $I_{N,p}(\gamma)$, $N = 1, 2, \ldots$ of shape measures. The new family $I_{N,p}(\gamma)$ includes an infinite number of measures which range over intervals wide enough to provide a discrimination capacity enough to distinguish among the shapes. The role of the parameter p is to provide tuning possibilities for the modified family and to expand the number of applications where the measures can be used efficiently.

A set of experimental results are provided in order to justify the theoretical considerations.

Key-words: Shape, shape descriptors, shape measures, object classification, object recognition.

I. Introduction

Shape is an object property which allows a wide spectrum of numerical characterisations. Such an object characterisation, by a set of assigned numerical values, is very convenient for a comparison of the objects and is very suitable for computer supported tasks like the object classification, identification or recognition. Many shape descriptors have been created so far. Some of them are very generic, like moment invariants [5] or Fourier descriptors [16], while the others relate to specific shape characteristics, e.g. rectilinearity [14], tortuosity [4], orientability [15], convexity [2], etc.

Another distinction among shape descriptors/measures can be based on the shape points used for their computation. In the past, most attention has been paid to the, so called, *area-based* shape measures. Such measures use all the shape points and, because of that, they are robust and suitable when working with low quality data or with low resolution images. On the other hand, due to a strong demand for very sensitive image processing tools, the *boundary-based* measures, which use the boundary points only, become more and more popular [6], [8], [13]. Notice that an additional demand for boundary-based shape descriptors/measures comes from the fact that some objects, like human signatures or letters, are linear in their nature and, because of that, area-based shape descriptors cannot be used for their analysis.

In this paper we consider a family of the boundary-based shape measures $I_N(\gamma) = \int_\gamma (x(s)^2 + y(s)^2)^N ds$ defined for any unit length curve γ, given in an arc-length parametrisation and with the boundary centroid coincident with the origin. Although this family consists of infinitely many shape measures, just a few of them can be used to distinguish among the shapes/objects. This is because the sequence $I_N(\gamma)$ converges quickly to 0, as $N \to \infty$. This follows from the here proved estimate $I_N(\gamma) \le 4^{-N}$, which holds independently of γ. In order to expand the number of measures which have a discriminative power big enough to be used in shape comparison based tasks, we have modified the family $I_N(\gamma)$. In addition we have introduced a "tuning" parameter p, or more precisely, we define a new family of the similarity invariant shape measures

$$I_{N,p}(\gamma) = \sqrt[N]{\int_\gamma (x(s)^2 + y(s)^2)^{p \cdot N} ds}.$$

Notice that the choice of p depends on the application where the modified descriptors $I_{N,p}(\gamma)$, are going to be used.

The paper is organised as follows. Section 2 introduces an infinite family shape measures and gives some related theoretical observations. Section 3 introduces a modified family of shape measures and gives an experimental justification that a use of shape measures from a modified family can be beneficial. Concluding remarks are in Section 4.

II. A Family of Shape Measures

In this paper we derive the main result of the paper. Throughout the paper we will use the following denotations:

- The Euclidean distance between points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ is denoted as $d_2(X, Y)$, i.e., as $d_2((x_1, x_2), (y_1, y_2))$.
- The centroid $\mathbf{C}_\gamma$ of a given curve γ, with length equal to 1, is

$$\mathbf{C}_\gamma = \left(\int_\gamma x(s) ds, \int_\gamma y(s) ds \right) \quad (1)$$

where the curve γ is given in an arc-length parametrisation $x = x(s),\ y = y(s),\ s \in [0, 1]$.

Also, even if not mentioned, all the curves γ are assumed:

- to be of a unit length;
- to be given in an arc-length parametrisation form: $x = x(s),\ y = y(s),\ s \in [0, 1]$, and
- to be positioned so that the centroid of γ and the origin coincide, i,e, $\mathbf{C}_\gamma = (0, 0)$.

We start with the following sequence/family of quantities which are invariant with respect to rotations, translations and scaling transformations:

$$I_1(\gamma) = \int_\gamma (x^2(s) + y^2(s))\ ds,$$

$$I_2(\gamma) = \int_\gamma (x^2(s) + y^2(s))^2\ ds,$$

$$\ldots$$

$$I_N(\gamma) = \int_\gamma (x^2(s) + y^2(s))^N ds.\ \ldots \qquad (2)$$

Indeed, $I_N(\gamma),\ N = 1, 2, \ldots$, are invariant with respect to rotations since the sub-integral functions $(x^2(s) + y^2(s))^N$ in $I_N(\gamma)$ are invariant with respect to rotations because they are based on the Euclidean distance of the curve points to the curve centroid. $I_N(\gamma)$ are invariant with respect to translations and scaling transformation by definition (prior to the computation the shape is positioned such that the shape centroid and the origin coincide and scaled such that the curve length is equal to 1).

Notice that $I_1(\gamma)$ is a boundary-based analogue [1] to the first Hu moment invariant [5]. Also, the function $r(s) = x^2(s) + y^2(s)$ which appears in the above invariants is the well known *radial function* which has already been used to analyze shape [3], [7].

The geometric meaning of the invariants $I_N(\gamma)$ is clear: It expresses the average value of the N-th power of the squared distance between the curve points and the curve centroid. Since different exponents N lead to a variable contribution $(x^2(s) + y^2(s))^N$ of points (x, y) from the curve γ, it makes sense to use several invariants $I_N(\gamma)$ together in the classification tasks. Such an approach, of using a family of invariants dependent on a "tuning" parameter, has been efficiently used in a recent work [12]. But the problem is that the invariants $I_N(\gamma)$, as given in (2), are not suitable for the classification tasks for large values of N. This is because $I_N(\gamma)$ converges quickly to 0 as N increases. Consequently, all values $I_N(\gamma)$ become almost the same (i.e., all equal to 0) for all N larger than a certain number N_0. This is the statement of the following theorem.

Theorem 1: Let γ be a planar curve with length equal to 1 and whose centroid coincides with the origin. Then

(a) $x^2 + y^2 \leq \frac{1}{4}$ for all $(x, y) \in \gamma$;

(b) $I_N(\gamma) \leq 4^{-N}$;

(c) $\lim_{N\to\infty} I_N(\gamma) = 0$.

Proof. First, we prove item *(a):* $x^2 + y^2 \leq 1/4$, i.e., γ lies inside a circle centred at the origin and whose radius is $1/2$. The items *(b)* and *(c)* are straightforward consequences of *(a)*.

Let γ be given in an arc length parametrisation $x = x(s),\ y = y(s),\ s \in [0, 1]$, and let the centroid $\mathbf{C}_\gamma$ of γ coincide with the origin. Also, let $A = (x(s_A), y(s_A)),\ s_A \in [0, 1]$, be a point belonging to the curve γ which has the greatest distance from the centroid $\mathbf{C}_\gamma$, and let

$$D_\gamma = d_2(A, \mathbf{C}_\gamma) = \max\{d_2(X, \mathbf{C}_\gamma) \mid X = (x, y) \in \gamma\}$$

denote this maximum distance. The point A splits the curve γ into two arcs, say γ_a and γ_b, defined as:

$$\gamma_a :\ \ x = x(s),\ y = y(s),\ \ s \in [0, s_A]$$

$$\gamma_b :\ \ x = x(s),\ y = y(s),\ \ s \in [s_A, 1]. \qquad (3)$$

Notice that there can be several points on the curve with this maximum distance from the centroid. In this case any of these points can be chosen as the point A.

Obviously the curve $\gamma = \gamma_a \cup \gamma_b$ lies inside a circle with radius $d_2(A, \mathbf{C}_\gamma)$ and centred at $\mathbf{C}_\gamma$. Now we chose the coordinate system such that

- $\mathbf{C}_\gamma$ coincides with the origin;
- the x-axis passed through the point A (and $\mathbf{C}_\gamma$, of course);

which implies that the curve γ lies in one of two half planes determined by the line $x = -D_\gamma$. Without loss of generality, we can assume that γ lies on the right side of the line $x = -D_\gamma$; i.e., all the points (different from A) belonging to γ have their abscissa value bigger than $-D_\gamma$ (as presented in Fig.1). Since the centroid $\mathbf{C}_\gamma = \left(\int_\gamma x(s)ds,\ \int_\gamma y(s)ds\right)$ coincides with the origin, we have $\int_\gamma x(s)ds = 0$ and further, we

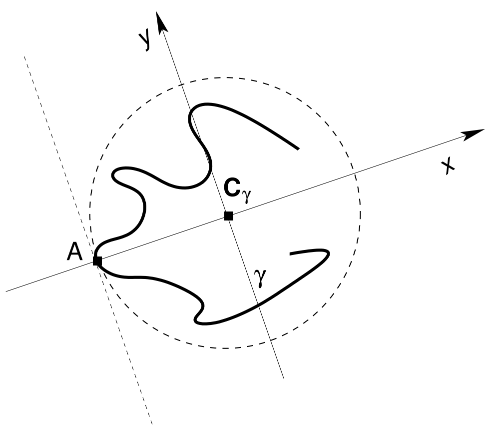

Fig. 1. $\mathbf{C}_\gamma$ is the centroid of γ and A is the point belonging to γ which has the greatest distance from $\mathbf{C}_\gamma$.

derive:

$$
\begin{aligned}
0 &= \int_\gamma x(s)ds \;=\; \int_{\gamma_a} x(s)ds \;+\; \int_{\gamma_b} x(s)ds \\
&= \int\limits_{s=0}^{s_A} x(s)ds \;+\; \int\limits_{s=s_A}^{1} x(s)ds \\
&\leq \int\limits_{s=0}^{s_A} (-D_\gamma + (s_A - s))ds \;+\; \int\limits_{s=s_A}^{1} (-D_\gamma + (s - s_A))ds \\
&= -D_\gamma \cdot s_A + (s_A)^2 - \frac{(s_A)^2}{2} \\
&\quad -D_\gamma \cdot (1 - s_A) + \frac{1}{2} - \frac{(s_A)^2}{2} - s_A \cdot (1 - s_A) \\
&= -D_\gamma + \frac{1}{2} - s_A \cdot (1 - s_A) \;\leq\; \frac{1}{2} - D_\gamma
\end{aligned} \tag{4}
$$

(**Note**: The following inequalities have been used:

$$x(s) \leq -D_\gamma + (s_A - s), \quad \text{for } s \in [0, s_A];$$

$$x(s) \leq -D_\gamma + (s - s_A), \quad \text{for } s \in [s_A, 1];$$

$$s_A \cdot (1 - s_A) \geq 0, \quad \text{for } s \in [0, 1].)$$

Thus, the just proven inequality $D_\gamma \leq \frac{1}{2}$ gives the required

$$x(s)^2 + y(s)^2 \;\leq\; \left(\max_{X\in\gamma}\{d_2(\mathbf{C}_\gamma, X)\}\right)^2 \;\leq\; D_\gamma^2 \;\leq\; \frac{1}{4}.$$

Now, by using the last estimate $x(s)^2 + y(s)^2 \leq 1/4$ we derive item *(b)* easily

$$I_N(\gamma) = \int\limits_{s=0}^{1} (x(s)^2 + y(s)^2)^N ds \leq \int\limits_{s=0}^{1} 4^{-N} ds = 4^{-N}.$$

Finally, $0 \leq I_N(\gamma) \leq 4^{-N}$ proves item *(c)* in the statement of the theorem. □

III. Experiments

Even though the family $I_N(\gamma)$ includes an unbounded number of shape measures, the statements of Theorem 1 show that only the first few elements of the family can be used for tasks which are based on shape comparison. This is because $I_N(\gamma)$ converges quickly to 0, i.e., $I_N(\gamma) = \mathcal{O}\left(4^{-N}\right)$ independently on the curve γ. So, after some N_0, the measures $I_N(\gamma)$, with $N > N_0$, are all very close to zero, and consequently do not have discriminatory potential necessary for an efficient comparison among the objects.

An immediate possibility to overcome such a problem is to consider the family $\sqrt[N]{I_N(\gamma)}$ instead the family $I_N(\gamma)$ and to get a new family with more elements which have a reasonably high discriminative capacity. An additional possibility is to introduce a tuning parameter $p \in (0, 1)$. We use both possibilities and define a new family of shape measures in the following way:

$$I_{1,p}(\gamma) = \int_\gamma (x^2(s) + y^2(s))^p \; ds,$$

$$I_{2,p}(\gamma) = \sqrt[2]{\int_\gamma (x^2(s) + y^2(s))^{p\cdot 2} \; ds},$$

$$\ldots$$

$$I_{N,p}(\gamma) = \sqrt[N]{\int_\gamma (x^2(s) + y^2(s))^{p\cdot N} ds}, \; \ldots \tag{5}$$

Under the assumptions that γ is scaled such that it has length equal to 1 and is positioned such that $\mathbf{C}_\gamma = (0, 0)$, it is easy to see that all the elements $I_{N,p}(\gamma)$ from the modified family are invariant with respect to similarity transformations. Also, the estimate $I_{N,p}(\gamma) \leq 4^{-p}$ can be proven analogously to the proof of Theorem 1.

Notice that if we select γ to be a circle, whose normalised arc length parametrization is

$$x = \frac{1}{2\pi}\cos(2\pi s), \quad y = \frac{1}{2\pi}\sin(2\pi s), \quad s \in [0, 1]$$

we obtain

$$
\begin{aligned}
I_{N,p}(\gamma) &= \sqrt[N]{\int_{s=0}^{1} (x^2(s) + y^2(s))^{p\cdot N} ds} \\
&= \sqrt[N]{\int_{s=0}^{1} \left(\frac{1}{4\pi^2}(\cos^2(2\pi s) + \sin^2(2\pi s))\right)^{p\cdot N} ds} \\
&= \sqrt[N]{\int_{s=0}^{1} \left(\frac{1}{4\pi^2}\right)^{p\cdot N} ds} \\
&= \left(\frac{1}{4\pi^2}\right)^p.
\end{aligned} \tag{6}
$$

The proven equality (6): $I_{N,p}(\gamma) = (4\pi^2)^{-p}$, for γ being a circle, implies that $I_{N,p}(\gamma)$ does not necessarily converge to zero. A further analysis would show even more: $I_{N,p}(\gamma) > 0$, for any curve γ whose length is equal to 1, and for all $N = 1, 2, \ldots$, and for all $p > 0$. In addition, it can be shown that $I_{N,p}(\gamma)$ can be arbitrarily close to 0 – i.e., more formally, for any $\varepsilon > 0$ there is a curve $\gamma = \gamma(\varepsilon)$ such that $0 < I_{N,p}(\gamma(\varepsilon)) < \varepsilon$.

So, we have constructed the family $I_{N,p}(\gamma)$ of an infinite number of shape measures/descriptors which all have an effective discrimination capacity (e.g., they do not converge to 0 as the measures from the initial family $I_N(\gamma)$ do). In addition, we have provided a "tuning" parameter p whose role is to control the behaviour of the shape measures from the family.

In the experiments which are performed in this section $p = 0.1$ has been selected.

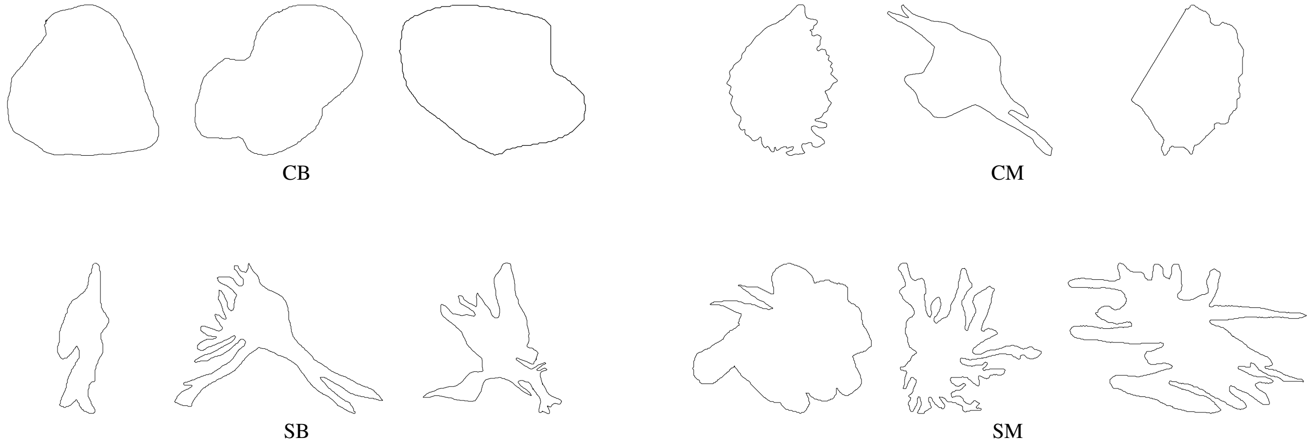

Fig. 2. Examples of the four classes of mammographic masses: circumscribed benign (CB), circumscribed malignant (CM), spiculated benign (SB), spiculated malignant (SM); they are drawn rescaled.

The invariants are demonstrated on a set of 54 masses from mammograms, combining images from the MIAS and Screen Test databases [9]. Some examples are shown in figure 2. Rangayyan *et al.* [9], [10] previously tested various shape measures on this data, classifying the masses as circumscribed/spiculated, benign/malignant, and CB/CM/SB/SM, in two group and four group classification experiments. They used the BMDP "7M" discriminant analysis program to carry out classification. Rosin [11] also used the same data to evaluate a variety of convexity measures for classification using a nearest neighbour classifier with Mahalanobis distances. The best results for these (both single descriptors/measures and combinations of shape descriptors/measures) are shown in table I.

Our new results are given in (table I). We have used $I_{N,p=0.1}(\gamma)$ for the classification. Table I shows that, when used individually, the proposed descriptors/measures are able to provide comparable classification accuracies to the existing methods in most cases. Moreover, when the best triple of the proposed descriptors/measures is applied for each of the classification tasks then the achieved accuracies exceed the previously used descriptors/measures.

IV. Concluding Remarks

This paper presents initial work on a use of infinite families of shape measures in object comparison based tasks including object classification, recognition or identification. We have started with a family of shape measures $I_N(\gamma)$ defined by the line integrals of the shape boundary radial function taken with an integer exponent. Such defined measures are invariant with respect to the scaling transformation and are defined for any integer N. Although, in theory, there are infinitely many of such measures, we have proven that most of these invariants are very close to zero and, because of that, in practice they cannot be used to distinguish among the shapes.

We have presented an opportunity to modify the family $I_N(\gamma)$ to obtain another family $I_{N,p}(\gamma)$ of shape measures which do not converge to 0, and consequently includes infinitely many measures which have a discrimination capacity high enough to be used for classification purposes. Measures from the family $I_{N,p}(\gamma)$ are also invariant with respect to the similarity transformations. There are other ways to modify the family $I_N(\gamma)$ and they will be investigated in the near future. For instance, one possibility is to replace the term $x^2(s) + y^2(s)$, in the sub-integral functions of $I_N(\gamma)$, by $k \cdot (x^2(s) + y^2(s))$, for a suitable k chosen from the interval $(-4, 4)$, but there are many more.

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	descriptors	circ/spic	benign/malig	4-way
1	compactness (C)	88.9	72.2	64.8
	Fourier shape factor (FF)	88.9	75.9	64.8
	$M_{c1\rightarrow c4}$, C, FF, MF_{1-3}	94.4	74.1	68.5
	C, FF	90.7	75.9	66.7
2	fractional convexity (f_{cv})	–	74	–
	spiculation index (SI)	–	79	–
	C, SI, f_{cv}	–	81	–
3	convexity ($C_{0.9}$ random)	94.4	59.3	57.4
	convexity (C_L)	85.2	74.1	68.5
4	$I_{N=10,p=0.1}$	88.9	70.4	64.8
	$I_{N=99,p=0.1}$	72.2	74.1	57.4
	$I_{N=10,p=0.1}$	88.9	70.4	64.8
	$I_{N=13,p=0.1}$, $I_{N=33,p=0.1}$, $I_{N=34,p=0,1}$	**96.3**	64.8	64.8
	$I_{N=120,p=0.1}$, $I_{N=133,p=0.1}$, $I_{N=147,p=0.1}$	70.9	**85.4**	60.0
	$I_{N=165,p=0.1}$, $I_{N=166,p=0.1}$, $I_{N=7,p=0.1}$	94.4	81.5	**79.6**

TABLE I

CLASSIFICATION RATES OBTAINED BY 1/ RANGAYYAN *et al.* [9] (NOTATION: $M_{c1\rightarrow c4}$ = CHORD LENGTH STATISTICS, MF_{1-3} = SHAPE FACTOR BASED ON LOW-ORDER MOMENTS), 2/ MUDIGONDA *et al.* [10], 3/ ROSIN [11] (NOTATION: $C_{0.9}$ RANDOM = THE PROPORTION OF RANDOM LINES THAT HAVE AT LEAST 0.9 OF THEIR LENGTH INSIDE THE SHAPE, C_L = CONVEXITY USING THE RATIO OF PERIMETERS OF THE SHAPE AND ITS CONVEX HULL), 4/ MEASURES PROPOSED IN THIS PAPER USING $p = 0.1$).

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