NEXUS VI

Architecture and Mathematics

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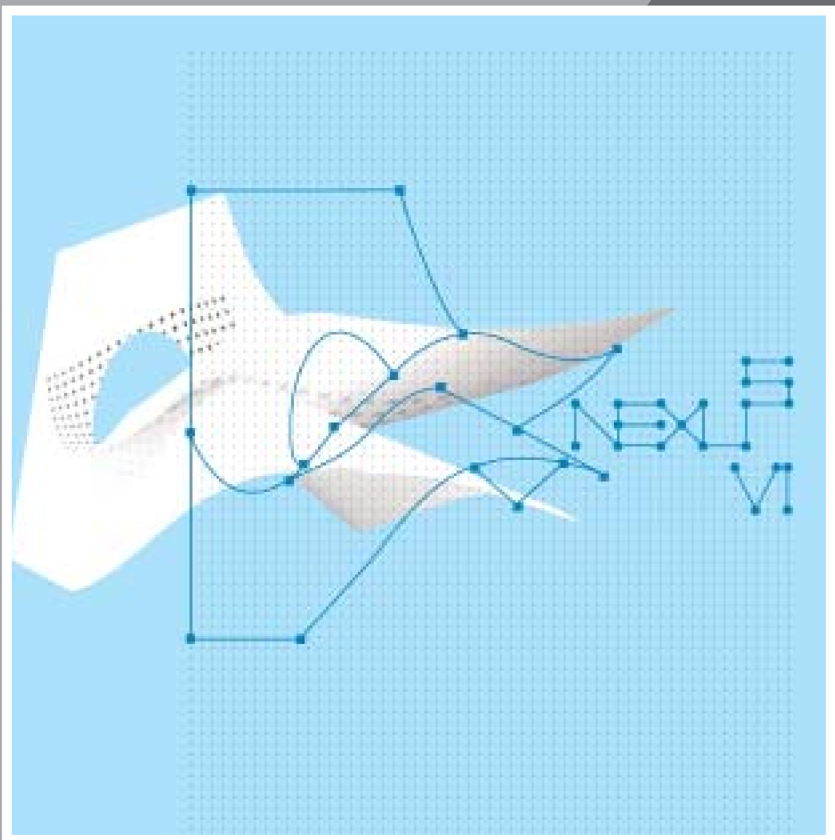

KIM WILLIAMS BOOKS

The compass, the ruler and the computer

Sylvie Duvernoy*
Paul L. Rosin°

Introduction

The purpose of research on design problems in historic architecture is usually to understand and to reveal the hidden theoretical sciences that the architects applied in their projects. The research may be oriented and supported by some written documents: architectural treatises from the epoch that provide the researcher with the basic information about the main theoretical design principles. Renaissance and early Baroque are periods in which written evidence of existing mature design theories are abundant. On the other hand, when studying ancient architecture from classical antiquity, apart from a few minor texts that incidentally mention various topics related to architecture, scholars have only the treatise *De Architectura Libri Decem* by Viruvius to help them understand the design principles from a theoretical standpoint. However, among the traditional typologies of Roman monuments, the amphitheatre is not among the case studies discussed by Vitruvius. This absence of scientific concern and of cultural recognition for this kind of building recently engaged the curiosity of modern scholars, who for a few years now have been trying to write, in Vitruvius's style, the missing chapters about amphitheatres.

The present research is a further attempt at revealing the intentional geometrical and arithmetical order that underlies the design layout of some Roman amphitheatres. The presence of this order may be proved either by using the same procedures that ancient designers probably used themselves – geometrical diagrams drawn with some kind of manual graphic device, relying on simple arithmetic calculations – or by applying modern analytical procedures provided by contemporary mathematics and computing.

In this paper both approaches will be discussed and will prove to be complementary.

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Of course, while applying modern mathematical tools, we do not assume that the ancient designers were aware of them; the objective is to demonstrate a knowledge, but not the knowledge of the tool used for the analysis itself. On the contrary, while applying a more traditional approach, which makes use of the basics of classical geometry and arithmetic, we consciously and deliberately imitate the supposed research methodologies of ancient designers, being careful to respect the limits of their contemporary knowledge, as much as this is possible.

The modern approach is based on the use of computer technology and specific software related to advanced mathematical theories, while the traditional approach is based on the manipulation of the compass, the ruler and the natural integers.

Here, two different analyses of the same monument – Pompeii's amphitheatre – were conducted in an alternating rather than in a parallel way. The results that each kind of analysis suggested then oriented and enriched the other aspect of the study. The conclusions do not come from the mere comparison of two independent researches but from the discussion of hypotheses that emerged alternatively from one or another aspect of the inquiry. Both analyses are based on the same data coming from an accurate survey of the monument that was conducted in 2001, according to the "polar methodology", i.e. by means of a single electronic theodolite, located in the approximate centre of the arena.

This study is the continuation of research already presented at the Nexus Conference in Óbidos (2002) [Duvernoy 2002] .

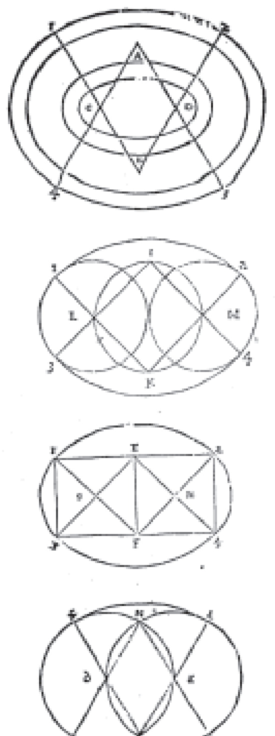


figure 1

Sebastiano Serlio's oval constructions (1537-1575)

1- Analysis through computer technology

Modern computing and mathematical facilities provide us with two extremely useful tools for the analysis of survey data: the first is a means of precisely fitting geometric models to the data and accurately measuring discrepancies between the model and data, and the second is statistical testing procedures to determine whether a given model is appropriate to the data. We will demonstrate the application of these tools to analyse the Pompeian amphitheatre.

Serlio's Oval Constructions. We already mentioned the lack of information about amphitheatres in Vitruvius's treatise. The main literary source for this kind of monument is Sebastiano Serlio's treatise on architecture [Hart and Hicks 1996]. Serlio assumes that amphitheatres were oval and he discusses oval shapes at some length. His four methods of construction are illustrated in figure 1; these have been used extensively in architecture over the years.

Only the first construction allows a variable aspect ratio; the other three are all fixed. It is the former we shall use in the following analysis. The oval consists of four circular arcs with centres ($\pm h$, 0) and (0, $\pm k$) and radii a-h and b+k respectively (fig. 2). It has the attractive property that the arcs join smoothly with tangent continuity. This geometric constraint can be algebraically expressed as:

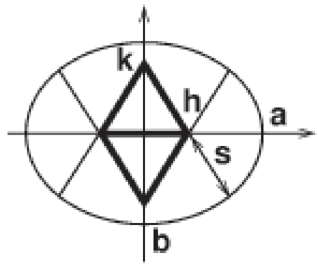


figure 2

Serlio's construction method for variable aspect ratio ovals.

$$h = \frac{k - \frac{a-b}{2}}{\frac{k}{a-b} - 1} \tag{1}$$

The construction can be generated conveniently using two equilateral triangles whose bases are centered on the origin. Their intersections with the axes determine h and k, which can be expressed as:

$$h = \frac{a-b}{\sqrt{3}-1}; k = \frac{\sqrt{3}(a-b)}{\sqrt{3}-1}.$$

The lengths of the circular arcs are specified by extending the triangles' diagonal sides by a length s. When s is increased both a and b are increased, and so it is not straightforward to choose geometrically correct values of h and s so as to achieve specified values of a and b. In fact, the ratio is given by

$$\frac{a}{b} = \frac{h+s}{h(2-\sqrt{3})+s}.$$

Not all aspect ratios can be achieved with this method since two arcs per quadrant are only drawn when s>0 such that

$$\frac{a}{b} \leq \frac{1}{2-\sqrt{3}} \approx 3.732 .$$

The ratio of the radii of the two circular arcs also varies, and can be shown to be

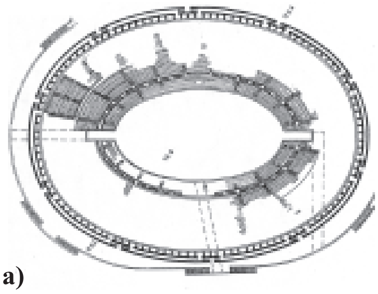
$$\frac{s}{2h+s}$$

Fitting the model to noisy data. The majority of the papers in the literature analyzing the layout of amphitheatres tend to overlay (often manually) the proposed shape (ellipse or oval) and then visually evaluate its appropriateness. While this might be feasible with perfect data (and perfect manual capability) the Roman amphitheatres have been subjected to damage over the years. On top of this, it is quite natural that inaccuracies and local adjustments would occur in construction. Thus, the positions of the walls no longer perfectly correspond to either an ellipse or an oval composed of circular segments.

This problem is aggravated by the similarity between an ellipse and an oval. Depending on the number of circular arcs used, an oval can be found that is a very good approximation of an ellipse [Rosin 1999], and so the difference between a four-centered oval and an ellipse only amounts to a few centimeters. Given such subtle differences, visual evaluation is inadequate, and a more objective approach is required. This has led researchers to perform a fit between the proposed models (i.e., ellipse, polycentric ovals) and use the fitting error as a criterion for selecting the most appropriate model [Trevisan 2000].

In this paper we also perform a best fit between the candidate models and the data. Rather than use a traditional least squares error term we minimize the mean absolute error using Powell's method [Press et al. 1990]. This is less sensitive to outliers than least squares. The error at each data point is taken as the shortest distance to the oval or ellipse. For the former the appropriate arc needs to be selected, making the fitting a non-linear, iterative procedure. In the case of the ellipse, computing the distances requires solving a quartic equation and choosing the shortest of the four solutions, and so it is also a non-linear process. (Note that to ensure proper testing of the ellipse we do not use the approximation to the distance to the ellipse developed earlier [Rosin 1998]. However, we found that the approximation was in fact very accurate, resulting in values in the following tables very close to those obtained using the true distance.) In addition to Serlio's oval, the best fitting four-centered oval that satisfies the tangent continuity constraint (Equation 1) is found. Since the oval and ellipse fitting is iterative an initial estimate is required; this is obtained by first directly fitting a conic with a non-linear constraint to guarantee an ellipse [Fitzgibbon et al. 1999].

Amphitheatre Data. Figure 3 shows the data acquired from the amphitheatre; the three nearly complete rings will be used for model fitting. Moving from the innermost to the outermost, they contain 99, 69, and 80 points. The ellipse and the two oval models were fitted to the data as described above, and the fits are



a)

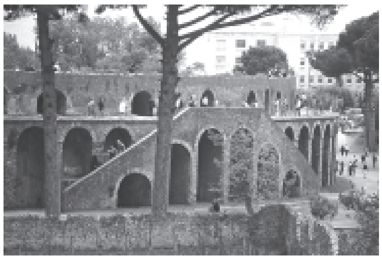
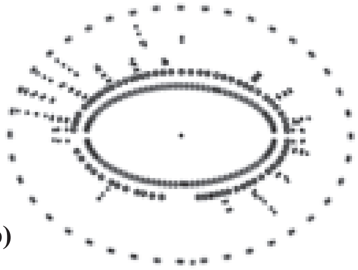


figure 4



b)

figure 3

figure 5

Fig. 3:
a) plan of the ruins of the monument,
b) measured data from the amphitheatre

Fig. 4: View from the outside façade

Fig. 5: View of the cavea from the arena.

illustrated in figure 6. It can be seen that all curves provide a very good fit to the data.

The mean absolute residual errors are given in Table 1. The best fits (lowest errors) are highlighted and it can be seen that the ellipse provides the best match to the data in all cases. However, despite the error values providing such objective and quantitative information, interpretation of these results must still be done with care. It is not sufficient to identify the best fitting model and declare that this was the one used in the planning and construction of the amphitheatre. For any set of data one of the models is bound to give a lower error than the remaining models. However, the question is whether that improvement in error of fit between one model and another is *statistically* significant.

ring	inner	middle	outer
ellipse	0.043	0.078	0.166
Serlio's oval	0.119	0.162	0.289
optimal oval	0.119	0.118	0.172

Table 1: Mean absolute errors of fitted models to the amphitheatre data.

Statistical Analysis. To analyze the data the first step is to determine what statistical model is appropriate. The fact that the different models have different degrees of freedom makes directly

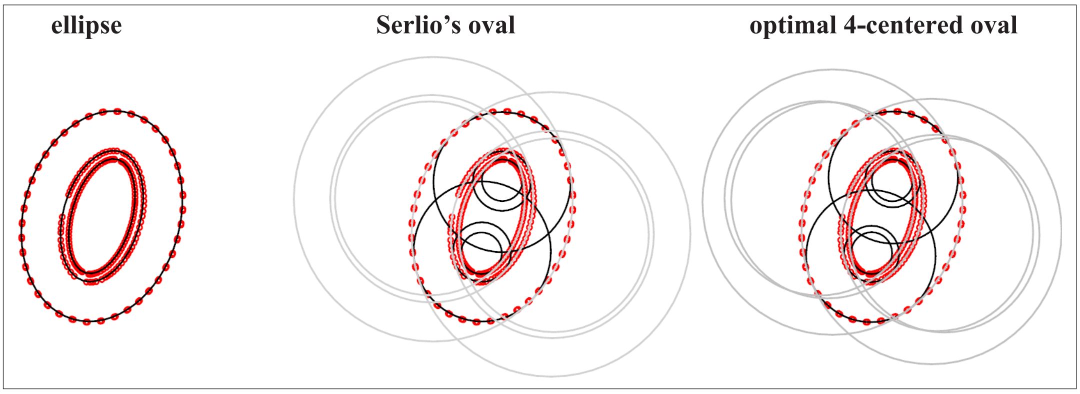


figure 6

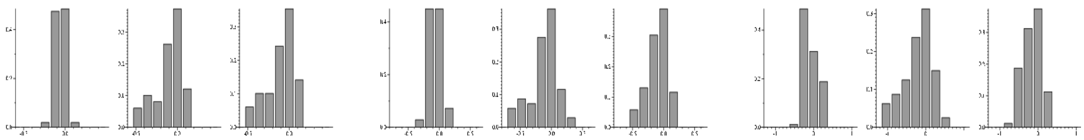


figure 7

Fig. 6: Best fit curves overlaid on the amphitheatre data

Fig. 7: Histograms of residuals for the three rings of points; each group of three shows the residuals for the fitted ellipse, Serlio’s oval, and the optimal oval

comparing them based on their mean fitting errors problematic. Instead we shall compare the models via their distributions of errors, and test if two error distributions are significantly different or not.

The simplest and most common model assumes normal distributions. Histograms of the *signed* residual errors for two of the rings are given in figure 7. While some histograms look reasonably normal others are doubtful, moreover their appearance is somewhat dependent on the histogram bin width. To check the hypothesis of normality more thoroughly the Shapiro-Wilk test [Siegal and Castellan 1988] is applied to the signed residuals (Table 2). If $p<0.05$ then the null hypothesis of normality is rejected, and thus few fits (indicated by the highlighted values) can be considered to have normally distributed residuals.

Table 2: Normal test computed by the Shapiro-Wilk statistic; only the highlighted values can be considered normal.

ring	inner	middle	outer
ellipse	0.211	0.493	0.003
Serlio’s oval	0.000	0.000	0.001
optimal oval	0.000	0.018	0.030

The above means that many standard statistical tests for comparing hypotheses are not applicable to this data. A non-parametric test is more appropriate since it does not require the assumption of a normal distribution. However, even when considering non-parametric tests, most still make some assumptions on the two groups of data (error distributions in our case) such as: i) they have the same standard deviations; ii) they have symmetric distributions; iii) the distributions have the same shape. There are few tests that do not assume any of the above, and so we are forced to use the median test, which tests if two independent groups differ in central tendencies (the alternative hypothesis) [Siegal and Castellan 1988]. The limitation of the median test is that since it makes so few assumptions it inevitably has a low statistical power, i.e. it needs more data points and/or lower noise levels to reach similar confidence levels than other comparable tests that make more assumptions (especially parametric tests).

ring	inner	middle	outer
Serlio's oval	7.733	1.378	0.461
optimal oval	6.761	0.000	1.842

Table 3: χ^2 values computed by the median test and applied to detect significant differences between the residuals from the ellipse fit and either of the two oval fits; only the highlighted values can be considered statistically significantly different

The results of the median test are shown in Table 3. If $\chi^2 \geq 6.64$ then the null hypothesis of no difference between the distributions (at a significance level α=0.01) is rejected. In the case of the inner ring there is sufficient information in the data to allow the median test to discriminate between the ellipse and ovals. For the other two rings, although the ellipse model also fits the data better than the four-centered ovals, it is not possible to prove that this is statistically significant.

It is of interest to compare a recently published analysis of a similar problem: testing whether patterns in prehistoric wall paintings correspond to geometric models (ellipses, spirals, polygons) [Papaodysseus et al. 2005]. In this case analysis was helped by two factors not available in our case of the Pompeii amphitheatre. First, the residual errors were shown to be normally distributed, and thus the more powerful t-test could be used to accept or reject the hypothesis. The second difference is that multiple instances of various patterns exist in the paintings. Assuming that the ones that appear to be circles were really intended to be painted as accurate circles enabled the painter's residual errors to be estimated. Consequently, it is straightforward to test if the residual errors incurred when fitting the other models (e.g. the ellipse) to patterns is greater than predicted by the error model, and should therefore be rejected.

Area and Perimeter Comparisons. Rather than concentrate on residual errors calculated from the data points Kimberling took another approach when analyzing the grassy square in Washington DC called the "Ellipse"[Kimberling 2004]. He compared values of area and perimeter measured from the data against the corresponding values determined from both an ellipse and tangent continuous oval. The ellipse was not fitted to all the data points, but rather its axis lengths were set to the same values as the diameter and width of the data. Likewise, the axis lengths of the oval were copied from the data. Two instances of the oval were then chosen by determining the values of *h* and *k* to match either the area or perimeter values measured from the data.

Table 4: Estimated values of area obtained from the models fitted to the amphitheatre data (in square meters).

ring	inner	middle	outer
polygon	*1862.47*	*2721.04*	*8746.07*
ellipse	1865.41	2729.57	8772.99
Serlio's oval	1870.11	2738.37	8797.88
optimal oval	1871.09	2734.41	8778.08

Table 5: Estimated values of perimeter obtained from the models fitted to the amphitheatre data (in meters).

ring	inner	middle	outer
polygon	*165.363*	*194.983*	*337.255*
ellipse	165.293	194.986	337.455
Serlio's oval	165.785	195.377	337.854
optimal oval	165.800	195.274	337.663

Table 4 and Table 5 show the values we have obtained for the amphitheatre. The "polygon" row indicates values derived from the original data, with adjacent points being connected to form a polygon. Calculating the remaining values is straightforward except for the perimeters of the ellipses, as no closed form solution is available; therefore this was computed numerically using Mathematica. It can be seen that in all cases the area and perimeter values measured from the data are most closely matched by the ellipses' values. However, there is insufficient information available to perform statistical analysis.

2- Analysis through ancient mathematics

By the means of these various computations we have made an analysis of ellipse and four-centered oval fits to the concentric walls of the Pompeii amphitheatre. We have shown that both models provide accurate matches to the data, but the ellipse always fits better. However, these differences are relatively small and subtle.

In order to come closer to a definite conclusion, the problem must be approached from a practical architectural standpoint and no longer from a purely theoretical point of view.

The geometry of the building. Pompeii is the oldest of the extant Roman amphitheatres. It belongs to a building type that relies mainly on ground modeling, rather than on heavy masonry technology. Its construction involved partly excavating and partly raising the natural ground. The level of the arena lies below the exterior ground level, while the upper part of the *cavea*, the tiered seating area, is higher. The stone seats for the spectators used to lie directly on the ground, pre-modeled to provide a suitable slope running all around the central void of the arena. Therefore, tracing an oval curve for the arena would have meant that two out of the four centres would have been placed high in the middle of the slope of the raised ground, much above the level of the arena itself, while tracing ellipses using "the gardener's method" (two poles planted in the ground, and a rope) meant working on a horizontal surface, and dealing with two "centres" only: the two focal points of the curve located symmetrically on its major axis, inside its perimeter.

The superimposition of the diagrams, either of the ovals and the ellipses, on the plan of the monument shows the hypothetical position of the centres of the circles and the foci of the ellipses inside the monument itself (fig. 8). While discussing the advantages

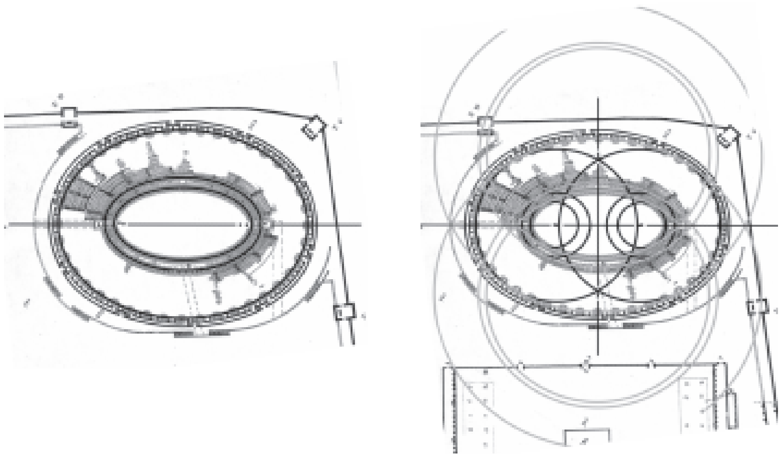


figure 8

The amphitheatre is built in the corner of the city wall.
The superimposition of the diagrams on the plan of the monument shows that the centres of the larger oval curves would be in an awkward position for an easy layout.

and disadvantages of each kind of curve, scholars always point out the fact that parallel ellipses cannot be traced from the same foci, while concentric oval curves can be traced from the same centres, and this very point obviously lead later architects to prefer the use of oval shapes for the design of later amphitheatres. But Pompeii's monument, being among the earliest buildings of this kind, has to be considered as innovative architecture, one that experimented with new geometrical patterns, and therefore attempting to reveal an utterly perfect diagram embedded in its composition might not be an appropriate goal (its builders did not even name it *amphitheatre*, but rather *spectacula*; the word "amphitheatre" itself appeared later on [Golvin 1988]). Figures 6 and 8 show that the hypothetical arcs of the ovals of Pompeii would not be concentric anyway and their centres would be so distant from one another that this could not be considered as errors due to historical traumas and subsequent permanent deformations.

In the late Roman Republic conic curves were part of common mathematical knowledge. We know that conic curves were first discovered by Manaechmus while searching for the solution to the Delian problem around 350 B.C. He is credited for having obtained them from the sectioning of acute-angled, right-angled and obtuse-angled cones. The curves were further studied by Aristaeus and Euclid. Sir Thomas Heath states that in Euclid's day some kind of focus-directrix property was already known [Heath 1981]. In any case, for tracing an ellipse only the knowledge of the existence of the foci is necessary (independent of the directrix). By the time Pompeii was built Archimedes and Apollonius of Perga had further investigated the properties of the conics. The ellipse was also known to Archimedes as the section of a cylinder; Apollonius even suggested that the planets had elliptic orbits of which the sun occupied one focus. In his treatise entitled "Conics", Apollonius examines various properties of the lines drawn from the foci to points on the curve or to specific points on straight lines tangent to the curve. And finally, proposition 52 of book III shows that if, in an ellipse, straight lines are deflected from the "*points resulting from the application*" (i.e., the foci) to any point on the curve, the sum of the distances will be equal to the main axis [Apollonius of Perga 1998]. This well-known property of the ellipse, which has been extensively applied to drawing elliptic curves by the so-called "gardener's method", because of its simplicity, is not emphasized by the author and no corollary of the proposition is provided to point out a reverse practical aspect.

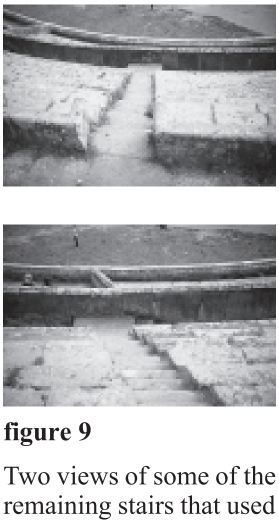


figure 9

Two views of some of the remaining stairs that used to divide the cavea into 40 different *cunei.*

Archimedes and Apollonius were the greatest mathematicians to dedicate themselves to the study of conics, at the close of the golden age of Greek geometry. They died respectively around 212 and 192 B.C., about a century and a half before the building of Pompeii's amphitheatre, which was not the first to appear in south Italy. We may be forgiven for thinking that we have here the proof of the interaction between science and art, and mathematics and architecture in the early Roman World.

Some attention must be paid as well to another aspect of the geometrical pattern of Pompeii, i.e. the radii that divide the cavea in several wedges (fig. 9). Some awkward attempts have already been made to determine the position of some converging points that would have facilitated their layout [Duvernoy 2002]. But in the context of an elliptical pattern and a non-oval shape, the wedges of the cavea must have been drawn in some other way than radial division from a centre that does not exist. The upper perimeter of the cavea is pierced by forty gates from which forty stairs used to go down to the *ima cavea*. The upper ellipse of the monument was thus divided into forty segments approximating the curve by an irregular forty sided polygon. If we do the same thing with the lower ellipse of the cavea, and if the vertices of the two polygons are connected by rays, the resulting diagram quite perfectly matches the data coming from the survey (fig. 10).

The inscription of polygons inside ellipses and circles is the methodology that Archimedes used to apply in order to find out simple ratios between the areas of ellipses and areas of circles having common "diameters" [Archimede 1974]. Archimedes is legendary for having attempted squaring the circle, but his efforts on squaring conics – the ellipse and especially the parabola – are less famous. Nevertheless, he stated that the area of a circle whose diameter is equal to the greater axis of an ellipse is in the same ratio to the area of this ellipse as its diameter is to the shorter axis of the ellipse (*Conoids and Spheroids*, prop. 4). Further propositions compare the areas of the ellipses to the areas of the rectangles that circumscribe them, and to circles and their circumscribing squares, etc. Since Archimedes also determined that the value of π can be approximated to 3+1/7, we may assume that the calculations about perimeters and areas of the various curves of Pompeii's amphitheatre, required for building construction and financial purchases (estimating material quantities and costs), were possible.

The arithmetic of the building: dimensions and numbers of Pompeii. The search for a module acting as the major common divisor of all dimensions [Duvernoy 2002] has shown that the axes of the three measured curves are respectively 10-19, 13-22 and 26-35 modules. Applying Archimedes' method for calculating the area, S=(3+1/7)ab, we find out that the area of the arena is exactly two thirds of the area of the curve closing the podium. In other words, the elliptic ring of the podium is exactly half the area of the arena. Because of the irrational value of π, numbers cannot be round for both the perimeter and the area of each curve, but ratios and proportions are precise.

curve	"diameters"		"radii"		perimeter	area
	A	B	a=A/2	b=B/2		
arena	10	19	5	9,5	45,5	150
podium	13	22	6,5	11	55	225
"hidden"	16	25	8	12,5	64,5	100 (3+1/7)
amphitheatre	26	35	13	17,5	96	715

Table 6: Examples of computation in modules and square modules. A "module" is equal to 12 Roman feet (roughly 3,50 meters) and a "square module" is 144 square Roman feet (roughly 12,25 square meters). Comparison between tables 4, 5 and 6 must be done considering the approximate value for π.

Table 6 includes numbers about the "hidden" curve that could not be measured since it lies beneath the cavea, but its position can be precisely assumed from the entrances to the *ambulacrum*, or passageway, behind the podium. This curve is the perimeter of the supporting wall of the outer part of the amphitheatre. It marks the division between the ground that was lowered and the ground that was raised: the "void" and the "solid". Therefore it must have been

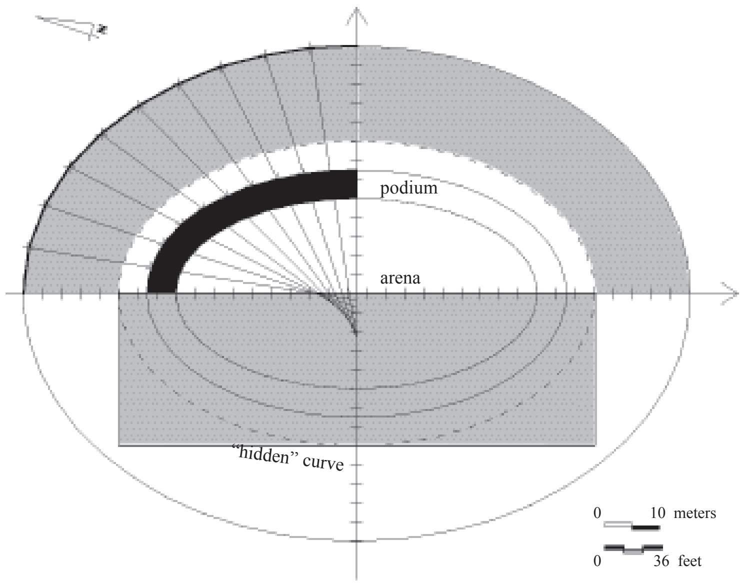


figure 10

Geometric diagram of Pompeii's amphitheatre. Highlighted areas (in grey) are of the same magnitude.

traced as a first step in the building process, since the supporting wall needed to be erected before starting any other operation. The area enclosed by the supporting wall, whose axes measure 16 and 25 modules, is exactly 314,2857 [=100(3+1/7)]. Consequently, the area of the raised cavea around the central void is

$$715 - 314{,}2857 \approx 400$$

which corresponds to the area of the rectangle circumscribing the "hidden" ellipse (16x25). In other words, the ratio between the "void" and the "solid" is equal to the ratio of the area of the ellipse and its circumscribing rectangle. The elliptic ring around the void has thus been "squared": 400 is not only a rectangular number but also a square one. The "squaring" of irregular areas has always been one of the main concerns of ancient mathematics since Egyptian times, and in the case of Pompeii's amphitheatre, the squaring of the elliptic ring has been achieved thanks to the clever choice of specific numbers for modular dimensions. Calculations and computations are therefore easy. Each of the 40 wedges of the cavea has an area of 10 square modules, etc.

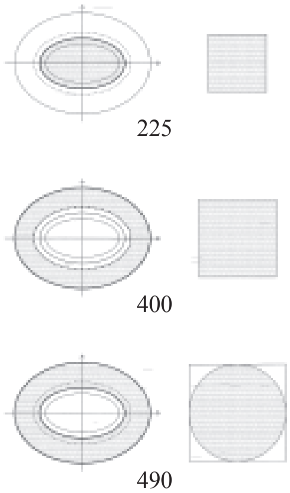


figure 11

The squaring of elliptic areas, and the figurate numbers of Pompeii's amphitheatre

Conclusion

The analysis of the amphitheatre of Pompeii by the means of ancient mathematics was thus accomplished from two different standpoints. First, by noting the curves' shape, their centres, and the tracing of the radii, we discussed the geometry (i.e., the manipulation of the classical drawing tools, straightedge and compass mainly), and then, by carefully interpreting the dimensions of the monument, thanks to our knowledge of ancient metrology, we discussed the arithmetic (i.e., the manipulation of the natural integers). The geometrical analysis and the arithmetical analysis both converge to the same conclusion. Furthermore they corroborate the conclusions suggested by the numerical analysis with modern mathematics (i.e., the manipulation of computer science). Therefore, the coherence of the results coming from our different approaches allows us to assert that the geometrical pattern of Pompeii's amphitheatre is a rare example of elliptic shape in architecture. Furthermore, its geometry and dimensions also show some of the finest evidence of direct application of the latest discoveries in mathematical knowledge and science in architectural design in classic antiquity.

And, most important, the influence of mathematical research on architectural formalism survived the geometric improvements or changes in successive amphitheatre design since the difference between an oval shape and an elliptic one is imperceptible to the observer.

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