Neural models of arbitrary chaotic systems: construction and the role of time delayed feedback in control and synchronization

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Abstract

This paper proposes a simple methodology to construct an iterative neural network which mimics a given chaotic time series. The methodology uses the Gamma test to identify a suitable (possibly irregular) embedding of the chaotic time series from which a one step predictive model may be constructed. A one-step predictive model is then constructed as a feedforward neural network trained using the BFGS method. This network is then iterated to produce a close approximation to the original chaotic dynamics.

We then show how the chaotic dynamics may be stabilized using time-delayed feedback. Delayed feedback is an attractive method of control because it has a very low computational overhead and is easy to integrate into hardware systems. It is also a plausible method for stabilization in biological neural systems.

Using delayed feedback control, which is activated in the presence of a stimulus, such networks can behave as an associative memory, in which the act of recognition corresponds to stabilization onto an unstable periodic orbit. Surprisingly we find that the response of such systems is remarkably robust in the presence of noise. We briefly investigate the stability of the proposed control method and show that whilst the control/synchronisation methods are not always stable in the classical sense they may instead be probabilistically locally stable.

We also show how two independent copies of such a chaotic iterative network may be synchronized using variations of the delayed feedback method. Although less biologically plausible, these techniques may have interesting applications in secure communications.

1 Introduction

This work draws its inspiration from [1]. On the basis of studies of the olfactory bulb of a rabbit Freeman suggested that in the 'rest state' the dynamics of this neural cluster is chaotic, but that when a familiar scent is presented the neural system rapidly simplifies its behaviour and the dynamics becomes more orderly, more nearly periodic than when in the rest state. We call this the 'retrieval behaviour' since it is analogous to the act of recognition. This suggests an interesting model of recognition in biological neural systems which is quite different from earlier attempts to use neural networks for pattern recognition or as associative memories.

To construct such a system we have to consider how best to construct neural models which exhibit chaotic dynamics. Neural network models which are dynamical systems are not (of course) new. The classical example is the Hopfield network [2], for which the simplest case considers nodes whose outputs are zero or one and where memories are associated with specified (preferably uncorrelated) point attractors. However, such a model cannot meet our needs. The state space is finite, consisting of fixed length vectors whose components are zero or one, and hence 'chaos'

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in the classical sense of dynamical systems, with its infinitely rich variety of modalities will never be exhibited. Indeed for a symmetric Hopfield network the dynamics are essentially trivial: starting from any initial state the network will simply iterate to a fixed point.

In contrast if the dynamics are chaotic then unstable periodic orbits are dense on the chaotic attractor and there are infinitely many of them. Thus an associative memory such as described by Freeman, for which the computations are performed to an arbitrary precision, could in principle accommodate infinitely many memories. At any rate such a system is not subject to the conventional Hopfield upper bound of $0.15n$, where n is number of neurons [3]. Of course, for the Hopfield net the situation is rather different. In the Hopfield model memories are associated with specified point attractors, whereas in the Freeman paradigm memories would be associated with unstable periodic behaviours which could not be specified ab initio. However, another attraction of the Freeman approach is that it introduces the possibility of responding to stimuli over varying time scales using behaviours with different periodicities.

Plainly we need to work with network models having continuous node outputs rather than the discrete outputs of the classical Hopfield model.

Chaotic dynamics have been observed in many artificial neural systems, either in continuous-time systems [4] or discrete-time systems [5]. An early example of a neural system which displays chaos at the neural level is the voltage-controlled oscillator neuron (or VCON) of [6]. This model, in contrast to all-or-none neuron models, generates voltage spikes that phase-lock to oscillatory stimulation, similar to the phase-locking of action potentials to oscillatory voltage stimulation observed in Hodgkin-Huxley preparations of squid axons [7].

In this paper, building on existing knowledge of smooth non-linear modelling techniques, we propose a methodology, suggested in part by Takens theorem [8], to build chaotic neural networks based on an iterative model of a conventional feedforward network trained to accurately model a given chaotic time series. We show how such networks can readily be constructed and give several examples, which are described in detail in the 'Supplementary materials and experimental results' html file associated with this paper – [SupMat].

2 Using the Gamma test to select an irregular embedding

The Gamma test was originally discussed in [9, 10]. It was subsequently used in synchronization [11], and control [12] of neural networks, and applied to feature selection for genetic sequence classification [13]. In [14] the Gamma test was elaborated for use in noise distribution reconstruction and used in modelling a river system. Finally a theoretical analysis and proof were given in [15].

If we consider measurements of input-output data collected from a *smooth* (unknown) process f we can write $y = f(x_1, \ldots, x_m)+r$ where $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ is the input vector, the scalar y is the associated output and r is a stochastic variable with mean zero which describes the noise on the measurement of the variable y. Despite the fact that f is unknown the Gamma test algorithm uses M such input-output pairs $(\boldsymbol{x}_i, y_i)$ $(1 \leq i \leq M)$ to obtain an estimate for the variance $\mathrm{Var}(r)$ in $O(M \log M)$ time. By examining different selections of input variables and finding that selection which produces an estimated noise variance closest to zero we can use the Gamma test as a feature selection algorithm.

One classical approach to modelling time series data is to construct the model by choosing the past values, up to some number m (often called the embedding dimension) to form the *inputs* of the model. The *output* is then the current value of the time series. The formal basis for this approach for dynamical system modelling was first studied by Takens [8].

Thus an *embedding* of a time series is a selection of past values which are used to predict the current value via a model constructed from the data. A *regular*, or full embedding, takes all past values up to some value m. A suitable value for m is often found by a technique known as the *false nearest neighbour* (FNN) algorithm [16], although we can also use the Gamma test for this purpose by computing Gamma values for progressively increasing m. Comparative experiments indicate that these two approaches seem to give very similar estimates for the embedding dimension, although in practice FNN seems somewhat faster.

An *irregular* embedding chooses some subset of the m past values, and there are $2^m - 1$ possible irregular embeddings once m is chosen. Provided m is not too large, say $m \leq 20$, we may search all possible irregular embeddings for one which gives a Gamma estimate closest to zero. This approach to locating suitable irregular embeddings was described with examples in [17]. The examples given in [SupMat] are also constructed in this way: we start from low or zero noise time series data, determine the embedding dimension, and then use the Gamma test approach to find a suitable irregular embedding. This irregular embedding is then used in the construction of a chaotic net as described in the next section.

3 A generic chaotic neural model architecture

Having identified a suitable embedding using the Gamma test we have illustrated in [SupMat] how the chaotic feedforward network is trained on the time series data using the BFGS algorithm [18], which provides progressive adjustment of the neural networks weights by gradient descent. This is a quasi-Newton method performed iteratively using successively improved approximations to the inverse Hessian, instead of the true inverse. The improved approximations are obtained from information generated during the gradient descent process.

A generic scheme for such a stimulus-response recurrent network is shown in Fig 1. The single output of the network feeds back into the inputs using delay buffers according to the irregular embedding previously determined by the Gamma test experiments. This embedding should contain enough information for predicting the next system state.

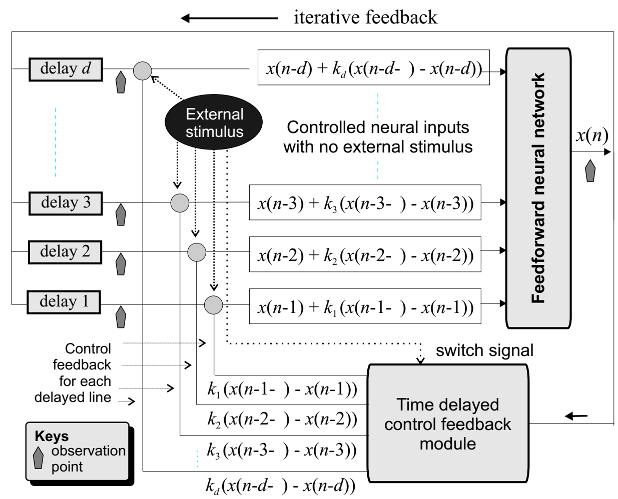

Figure 1: Generic chaotic neural net scheme.

4 Controlling neural network

Having seen how to exhibit neural chaos the next question becomes how to control it? We originally approached this issue [19] by considering a range of existing techniques which since 1989 have been used to control an enormous diversity of chaotic systems. Historically the first of these was a technique due to Ott, Grebogi and Yorke, known as the OGY method [20]. The basic idea is that a chaotic system exhibits numerous unstable periodic orbits and, having located one such behaviour, the OGY method seeks to stabilize this orbit using small variations of some accessible system parameter.

Most such methods require careful and systematic analysis of the chaotic dynamical behaviour and prior specification of the target unstable fixed point, which is difficult and computationally expensive, before successful control can be achieved. Moreover, such control techniques are external to the system being controlled, whereas for a neural system to behave as described by [1] the control should be intrinsic to the neural dynamics. Nevertheless, such preliminary studies served as a useful starting point for studying the control of neural chaos. A simple example of controlling a chaotic artificial neural network using such techniques is given in [21].

The dynamics of large neural ensembles are high-dimensional, and whilst the OGY technique is an effective tool for the control of low dimensional chaos it needs further elaboration for effective control of higher dimensional systems. Indeed, for higher dimensional systems it may be that other types of control procedures will prove far more effective.

To investigate the control of higher dimensional chaos we examined a well studied dynamical system described by a modified form of the Euler equations, the chaotic satellite attitude control problem. In [22] we apply various techniques to a variation of the chaotic satellite attitude control problem and show that it is possible to stabilize the system in a situation where five of the six sensors (three angular velocities and three attitude angles) and two of the three thrusters are inoperative. It emerged from the work of that paper that a remarkably simple and effective method of stabilization onto an unstable periodic behaviour can be effected by the application of delayed feedback. Delayed feedback to control continuous dynamical systems exhibiting chaos was first suggested in [23]. We use a modified version of this approach to stabilize an iterative neural model (previously trained to generate chaotic behaviour in the 'rest state') in the presence of an input stimulus.

One of the attractions of delayed feedback stabilization is that it has a very low computational overhead and so is extremely easy to implement in hardware. It would also be very easy to implement in biological neural circuitry and so offers one plausible mechanism whereby such stabilization might occur.

We illustrate this process using the previously constructed examples. Again the experimental details are given in [SupMat].

For stabilization control a multiple (gain constant) of the delayed feedback is added to each neural network input specified by the irregular embedding, based on the idea from Pyragas' delayed feedback control. The control module is shown in Fig 1 and the control perturbation for the i^{th} input at the n^{th} iteration is $k_i(x_i(n-i-\tau) - x_i(n-i))$, where k_i is a gain constant and τ is the delay time.

We imagine that the presence of an external stimulus excites (activates) the control circuitry, which is otherwise inhibited. Thus to achieve a stabilized dynamical regime in response to a stimulus the control is switched on at the same time as the external signal is fed into the input line x_n. By varying the external signal in small steps and holding the new setting fixed long enough for the system to stabilize we can observe the response of the network to small changes in stimulus.

In the diagram, τ is the same for each control perturbation but of course, we could set τ to be different on each control line. External stimulus of the network can be applied to the controlled inputs as shown in the diagram. The control module should switch on automatically and simultaneously whenever there is an external stimulation. Variations of stimulation, such as on the control delayed feedback lines may also be used.

We determine that the response to a particular stimulus is remarkably robust in the face of noise. A result which we found to be rather surprising whilst at the time extremely encouraging.

The particular unstable periodic orbit which is stabilized depends quite strongly on the precise character of the applied stimulus. Thus the system can act as an associative memory in which the act of recognition corresponds to stabilizing onto an unstable periodic orbit which is characteristic of the applied stimulus. The entire artificial system therefore exhibits an overall behaviour and response to stimulus which precisely parallels the biological neural behaviour observed by Freeman.

5 Synchronization

The idea of synchronising two independent copies of identical chaotic dynamical systems has been of increasing recent interest. Results shown by Skarda and Freeman [24] support the hypothesis that neural dynamics are heavily dependent on chaotic activity. Nowadays it is believed that synchronization plays a crucial role in information processing in living organisms and could also lead to important applications in speech and image processing [25]. Moreover, due to the important role that secure communications plays in industrial and banking communications, the po-

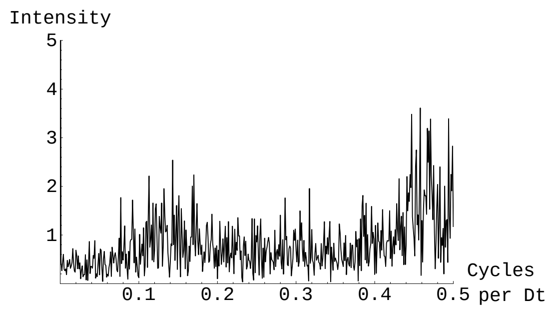

Figure 2: FFT of the Hénon map time series training data.

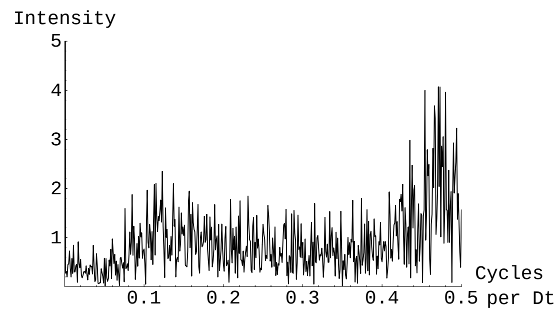

Figure 3: FFT of the Hénon map trained neural network (cycles per iteration).

tential application of chaotic synchronization to secure communications is receiving increased attention.

We show how two identical copies of a chaotic neural system may be synchronized using two variations of the method used for control. The methods used here for synchronisation are discrete versions of a variation of Pyragas' method but are not identical to the method for stabilization. The second of these methods was first demonstrated in [11].

Whilst less biologically plausible than our method of control, the synchronisation techniques may possibly have interesting applications. As things stand we do not claim that using chaotic carriers as an encryption technique is a particularly secure method but, if one is intending to use such encryption techniques, it is our belief that the use of chaotic neural networks may offer several advantages. Using the methods described in the present paper a neural network can be trained on virtually any map, for example a hyperchaotic system which has a more complex time signal (and might therefore be expected to be more difficult for a third party to synchronize). Iterated using hardware such neural networks could provide very high frequency chaotic carriers and the feedback methods of control have the advantage of being extremely easy to integrate into such a hardware implementation.

One can imagine having implemented a neural network with fixed input, hidden, and output layers in hardware and just by downloading a new set of weights having a different chaotic signal to act as a carrier. This also offers another potential advantage, since the chaotic map can readily be changed in such a system the effective 'noise' spectrum of the carrier can easily be manipulated to have particular (desirable) characteristics without modification of the underlying hardware. Fig 2 illustrates the FFT ($M = 1000$) of the training data for the Hénon map and can be compared with Fig 3 the FFT for the output of the iterated trained Hénon network, which is the amplitude of the transmitted carrier.

6 Conclusions

We have outlined a systematic methodology which reduces the construction of an iterative neural network, which closely models time series data from a given chaotic system, to a relatively automatic process. This construction method is based on finding a suitable (usually irregular) embedding using the Gamma test and has been illustrated in [SupMat] using several examples. The only decisions which remain to be made in the construction process are the neural network architecture and the actual training algorithm. The architecture will be determined by the complexity of the surface being modelled. Since we know from the Gamma test an estimate for the MSError we can start with a simple network and increase the number of hidden nodes until the training algorithm reaches the required target error. We have found that in practice a slightly modified form of the BFGS algorithm produces quite rapid convergence to the target MSError and compares very favourably with other forms of gradient descent.

Prompted by the work of Freeman [1] we are interested in biologically plausible mechanisms of stabilization for chaotic neural networks and have illustrated that all the examples considered can be stabilized via time-delayed feedback onto to unstable periodic attractors which are to a large degree characteristic of the applied stimulus. Moreover we have found that these behaviours are relatively robust in the face of noise. We conclude that if stabilization of chaos is significant in biological information processing then time delayed feedback is one possible mechanism by which this might be achieved. However, one should note that in the model we have described the response of the system to a particular stimulus cannot be specified *ab initio* and it remains an open question as to how such a system might be encouraged to learn particular responses without destroying the features of interest (such as the chaotic behaviour).

Another aspect of this model is that since the response may be a high-order periodic behaviour we could imagine a system which responds in a continuing way over a longer time scale. Thus, for example, the smell of a known predator could trigger a gait circuit and produce an evasion behaviour as long as the stimulus remained present.

In a similar way it is conjectured that synchronisation of different neural clusters may also play an important role in biological information processing. Again we have shown that synchronisation of two identical chaotic networks can be achieved very easily by means of time delayed feedback. We have illustrated using several examples two possible ways of accomplishing this.

Finally, irrespective of the interest of chaotic neural synchronisation in biological information processing there is the potential application of these ideas to secure communications. We have taken one of the synchronisation examples and illustrated how a simply encoded binary message may be masked by the chaotic neural carrier and then transmitted and recovered by an identical neural system at the receiver. As discussed in the introduction if chaotic carriers were to prove viable as a method of encryption then the use of neural networks may offer several advantages. In particular, to switch from one chaotic carrier to another could be done without having to reconfigure the underlying hardware.

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