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Anisotropic porous penetrative convection

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Abstract

A linear instability analysis and a nonlinear energy stability analysis is developed for convection in an anisotropic porous medium. The nonlinear analysis is very important since a standard energy method does not in the present situation yield unconditional stability and a weighted analysis must be employed to yield global results, and in addition the nonlinear energy results yield a valuable threshold indicating where possible subcritical instabilities may form. Significantly, we find that when a quadratic density temperature law is used in the anisotropic convection model of Tyvand & Storesletten (1991), then the growth rate σ is *always* complex provided we are in the anisotropic situation. Thus, the nature of bifurcation into convection is very different from the Boussinesq situation and is always via an oscillatory instability.

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1. Introduction

The subject of thermal and solutal convection in porous media has been receiving much recent attention; in particular, the effects of anisotropy in the porous matrix, boundary layers due to a Brinkman effect, and nonlinearity due to a Forchheimer contribution have been under scrutiny, see e.g. Qin & Kaloni (1992,1993,1994), Qin *et al.* (1994), Tyvand & Storesletten (1991), Straughan (1993), and the references therein. In Tyvand & Storesletten (1991) the first study was performed of the determination of when convection will begin in an anisotropic porous medium with oblique principal axes. These writers studied a porous medium in which the permeability is transversely isotropic with respect to an axis which makes an angle of β to the horizontal. The angle β may have an arbitrary value between 0 and 90° and Tyvand & Storesletten found many differences with convection in a classical isotropic porous medium; for instance, they observed that whereas in the isotropic situation the preferred motion at the onset of convection is in the form of rolls with square cross sections, when the medium is anisotropic the preferred motion may well be in tilted cells. The analysis of Tyvand & Storesletten (1991) was based on a linearised instability procedure.

The present paper takes the configuration of Tyvand & Storesletten (1991) but allows for a non-Boussinesq density in the buoyancy force. This is a physically important change since many geophysical problems involve convection with water near the freezing point, cf. Nield & Bejan (1992), George *et al.* (1989), Payne *et al.* (1988), Qin *et al.* (1994), Straughan (1993), and in this situation penetrative convection may develop and a quadratic density law must be employed. In practice, there will be anisotropy present in convection in a porous medium and the buoyancy density will not be linear and so we believe our results are relevant to other real life situations; anisotropy is often found when soil lofting has occurred due to frost heave, cf. George *et al.* (1989), and other anisotropic effects are discussed by Nield & Bejan (1992), particularly pages 178–180 and 229–231.

While we concentrate mainly on a quadratic temperature dependence for the density, we could easily generalize this. However, we briefly investigate another model for penetrative convection in an anisotropic porous medium, namely one which includes the penetration effect via a heat supply, cf. Roberts (1967), Matthews (1988). For the applications we have in mind, we believe the quadratic density law is more appropriate, but nevertheless, we observe that the significant findings of our work are also present for the heat source model.

We perform a complete linear instability analysis; however, this only gives information on instability and the onset of convection. It does not preclude the possibility of subcritical bifurcation. Hence, we also develop a nonlinear energy stability analysis. This analysis yields global three - dimensional nonlinear stability bounds and as these turn out to be very close to those obtained in the linear analysis we can be sure that the linear method has captured the essential physics of the onset of convection.

Tyvand & Storesletten (1991) find that at the onset of convection the growth rate σ is real and so convection commences via a stationary convection mechanism. For penetrative convection with a quadratic density model in the *isotropic* case one can also show $\sigma \in \mathbf{R}$. However, we here find that any departure from isotropy produces a complex growth rate and so the convection process always begins via an oscillatory convection mechanism. We believe this feature is very novel and interesting; certainly, it means a nonlinear energy stability analysis is more necessary since in other convection problems where overstability occurs the energy method has usually led to somewhat weak results suggesting possibily large areas of subcritical bifurcation may be present. We shall also show that the complexity is not due to an anisotropic permeability or thermal conductivity alone, even if these quantities are allowed to be functions of temperature. The complex growth rate is due to inhomogeneity introduced by the penetrative convection coupled with the effect of the porous medium having an anisotropic permeability.

The plan of the paper is as follows. In the next section we develop the basic equations for penetrative convection in an anisotropic porous medium for which the permeability is transversely isotropic with respect to an inclined axis. We then explain how we investigate linear instability numerically. After this we show how the energy method may be applied to delimit the range of Rayleigh numbers where subcritical instabilities may possibly arise. In the penultimate section, we briefly examine an internal heat source model for penetrative convection in an anisotropic porous medium. Finally, we include several numerical results and interpret our findings.

2. The Model

We consider a fluid saturated porous layer contained between horizontal planes $z = 0, d$ with gravity in the negative z direction; the z- direction is denoted by the vector $\mathbf{k}$, with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ being the standard Cartesian basis. The upper plane is maintained at a temperature $T_1(> 0°C)$ and the lower plane remains at temperature $0°C$. In this region the density of water is modelled well by the relation

$$\rho(T) = \rho_m \left[1 - \gamma(T - T_m)^2\right], \tag{2.1}$$

where $\rho_m = \rho_4$ is the density of water at $T_m = 4°C$, T is temperature, and $\gamma(\approx 7.68 \times 10^{-6}(°\mathrm{C}^{-2})$ is the appropriate expansion coefficient, Veronis (1963). Of course, equation (2.1) allows for a density maximum at $4°C$ and hence if $T_1 > 4$ then the lower part of the layer has a tendency to be gravitationally unstable whereas the upper part is stable. In this situation the phenomenon of penetrative convection can arise where the fluid begins to convect in the lower layer and induces convective motion in the upper, cf. Veronis (1963), Nield & Bejan (1992), Straughan (1992,1993).

The permeability tensor $\mathbf{K}^*$ is selected as in Tyvand & Storesletten (1991), namely,

$$\mathbf{K}^* = K_{\|}\mathbf{i}'\mathbf{i}' + K_{\perp}(\mathbf{j}'\mathbf{j}' + \mathbf{k}'\mathbf{k}') \tag{2.2}$$

where $K_{\|}$ and $K_{\perp}$ are the longitudinal and transverse components of permeability. The plane of the porous layer is that for which $\mathbf{k}$ is orthogonal to the layer and $\mathbf{i}$ is aligned with the projection of $\mathbf{i}'$ on the (x, y) plane. The angle β is the angle between the vectors $\mathbf{i}$ and $\mathbf{i}'$. The inverse permeability tensor $\mathbf{M}$ satisfies

$$\mathbf{M}.\mathbf{K}^* = K_{\perp}\mathbf{I}, \tag{2.3}$$

$\mathbf{I}$ being the identity. Then (Tyvand & Storesletten (1991))

$$\begin{aligned}\mathbf{M} =& \xi\mathbf{i}'\mathbf{i}' + \mathbf{j}'\mathbf{j}' + \mathbf{k}'\mathbf{k}' \\ =& M_{11}\mathbf{ii} + M_{13}(\mathbf{ik} + \mathbf{ki}) + M_{33}\mathbf{kk} + \mathbf{jj}\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}M_{11} =& \xi\cos^2\beta + \sin^2\beta \\ M_{13} =& (\xi - 1)\cos\beta\sin\beta \\ M_{33} =& \cos^2\beta + \xi\sin^2\beta,\end{aligned} \tag{2.5}$$

where $\xi = K_{\perp}/K_{\|}$ is the anisotropy parameter.

The fluid motion in the porous layer then satisfies a Darcy law of form

$$\mathbf{K}^*\nabla p = -\mu\mathbf{v} - g\mathbf{K}^*\mathbf{k}(2\rho_m\gamma T_m T - \gamma\rho_m T^2), \tag{2.6}$$

where p is the pressure (modified by the constant term arising from (2.1)), μ viscosity, $\mathbf{v}$ velocity, and $\mathbf{k} = (0, 0, 1)$. In terms of the inverse permeability tensor $\mathbf{M}$ equation (2.6) may be rewritten and we group the resulting equation together with the incompressibility condition and the equation of energy balance which governs the evolutionary behaviour of the temperature field to yield the following system of governing equations

$$p_{,i} = -\frac{\mu}{K_{\perp}} M_{ij}v_j - 2\rho_m\gamma T_m g k_i T + \gamma\rho_m g k_i T^2, \tag{2.7}$$

$$v_{i,i} = 0, \tag{2.8}$$

$$T_{,t} + v_i T_{,i} = \kappa\Delta T, \tag{2.9}$$

where κ is the thermal diffusivity which for the most part we treat as constant as in Tyvand & Storesletten (1991), and where standard Cartesian indicial notation together with the Einstein summation convention is employed.

The basic solution to (2.7) - (2.9) in whose stability we are interested is

$$\bar{v}_i = 0, \qquad \bar{T} = \hat{\beta} z, \tag{2.10}$$

where

$$\hat{\beta} = T_1/d,$$

and where the hydrostatic pressure $\bar{p}$ may be found from (2.7). To study the (nonlinear and linear) stability of (2.10) we introduce perturbations (u_i, θ, π) to (v_i, T, p) by

$$v_i = \bar{v}_i + u_i, \qquad T = \bar{T} + \theta, \qquad p = \bar{p} + \pi$$

and we then non-dimensionalize the resulting equations for the perturbations with the scalings $t^*, U, T^\sharp, P$, being typical time, velocity scale, temperature, pressure, respectively,

$$t^* = d^2/\kappa, \qquad U = \kappa/d, \qquad \mathbf{x} = \mathbf{x}^* d,$$
$$T^\sharp = \left(\frac{\beta\mu\kappa}{\rho_m \gamma g T_1 K_\perp}\right)^{1/2}, \qquad P = \frac{\mu\kappa}{K_\perp}, \qquad R = T_1 \left(\frac{d K_\perp \rho_m \gamma g}{\mu\kappa}\right)^{1/2},$$

where R^2 is the Rayleigh number, and a penetration parameter $\hat{\xi}$ appears where

$$\hat{\xi} = T_m/T_1 \quad (= 1/\lambda)$$

where λ is Veronis' (1963) parameter. In this scaling the non-dimensional equations for the perturbations become

$$\pi_{,i} = -M_{ij} u_j - 2R(\hat{\xi} - z) k_i \theta + k_i \theta^2, \tag{2.11}$$
$$u_{i,i} = 0, \tag{2.12}$$
$$\theta_{,t} + u_i \theta_{,i} = -Rw + \Delta\theta, \tag{2.13}$$

where $w = u_3$.

To these equations boundary conditions must be added. These are

$$w = \theta = 0 \qquad \text{at} \qquad z = 0, 1, \tag{2.14}$$

although an upper boundary condition more appropriate to patterned ground formation (George *et al.* (1989)) is considered in section 5. In addition the solution satisfies a plane tiling shape in the x, y directions which allows for periodicity in these directions but also is capable of describing more than rectangular shapes, such as hexagons, see the detailed discussion in Drazin & Reid (1981), pp. 52 - 59. The three - dimensional periodicity cell which results will be denoted by V.

3. Linearized instability and the principle of exchange of stabilities

The equations governing linearized instability are obtained from (2.11) - (2.13) by neglecting the convective nonlinearity in (2.13) and the θ^2 term in (2.11) so that the linearized system is, after assuming a temporal growth rate like $e^{\sigma t}$,

$$\pi_{,i} = -M_{ij}u_j - 2RMk_i\theta, \tag{3.1}$$

$$u_{i,i} = 0, \tag{3.2}$$

$$\sigma\theta = -Rw + \Delta\theta. \tag{3.3}$$

We shall say the *strong form* of the principle of exchange of stabilities holds if $\sigma \in \mathbf{R}$.

By a method entirely different from that considered here Tyvand & Storesletten (1991) argue that $\sigma \in \mathbf{R}$ in the equivalent anisotropic problem without penetrative convection; the difference then is that (3.1) - (3.3) are replaced by

$$\pi_{,i} = -M_{ij}u_j + Rk_i\theta,$$
$$u_{i,i} = 0,$$
$$\sigma\theta = Rw + \Delta\theta.$$

We shall also show $\sigma \in \mathbf{R}$ but we momentarily allow an anisotropic, but symmetric thermal conductivity, and we may also allow the viscosity μ to depend on temperature T (in an arbitrary way) and the thermal conductivity κ_{ij} to likewise depend on T; studies of convection with $\mu(T)$ have attracted much recent attention, see e.g. Or (1989), Qin *et al.* (1994), Richardson & Straughan (1993), and the references therein. In this case the *linearized* system for convection in an anisotropic porous medium *without penetrative convection* is

$$\pi_{,i} = -M_{ij}(z)u_j + Rk_i\theta, \tag{3.4}$$

$$u_{i,i} = 0, \tag{3.5}$$

$$\sigma\theta = Rw + \big(\kappa_{ij}(z)\theta_{,j}\big)_{,i}. \tag{3.6}$$

To demonstrate $\sigma \in \mathbf{R}$ the procedure is as follows. Multiply (3.4) by the complex conjugate u_i^*, (3.6) by θ^* and integrate over the cell V. There results

$$0 = -< M_{ij} u_i^* u_j > + R < \theta w^* >,$$

$$\sigma \|\theta\|^2 = R < w\theta^* > - < \kappa_{ij} \theta^*_{,i} \theta_{,j} >,$$

where $< \cdot >$ denotes integration over V and $\| \cdot \|$ is the norm on the (presently complex Hilbert space) $L^2(V)$. We add the above two equations and since M_{ij} and κ_{ij} are both symmetric we may then take the imaginary part of the sum to deduce, if $\sigma = \sigma_r + i\sigma_1$

$$\sigma_1 \|\theta\|^2 = 0.$$

Since $\theta \neq 0$ we must have $\sigma_1 = 0$ and so $\sigma \in \mathbf{R}$. Thus, for the non-penetrative convection case in which ρ is a linear function of T, the strong form of the principle of exchange of stabilities holds even if both permeability and thermal diffusivity are symmetric but anisotropic functions of temperature.

Suppose now we are in the *isotropic* situation but *with* penetrative convection, so that (3.1) - (3.3) hold although now $M_{ij} = \delta_{ij}$. By taking the operator curlcurl of (3.1) and the third component of the result we find

$$0 = \Delta w + 2RM\Delta^* \theta, \tag{3.7}$$

where Δ^* is the horizontal Laplacian, i.e.

$$\Delta^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,.$$

Next, eliminate w between (3.7) and (3.3) to deduce

$$\sigma \Delta \theta = \Delta^2 \theta - 2R^2 M \Delta^* \theta. \tag{3.8}$$

We now note that thanks to (3.3), $\Delta\theta = 0$ also on $z = 0, 1$, and we may then multiply (3.8) by θ^* and integrate over V to find

$$-\sigma \|\nabla\theta\|^2 = \|\Delta\theta\|^2 + 2R^2 < M|\nabla^*\theta|^2 >, \tag{3.9}$$

where $\nabla^* = (\partial/\partial x, \partial/\partial y)$. Upon taking the imaginary part of this expression we conclude

$$\sigma_1 \|\nabla\theta\|^2 = 0,$$

and hence $\sigma \in \mathbf{R}$.

We stress that *both* of the above methods *fail* when applied to (3.1) - (3.3) in the *anisotropic* case, i.e. when $\beta \neq 90°$, $\xi \neq 1$.

It is instructive to follow an analogous method to that leading to (3.8) to see what happens. We may eliminate the pressure perturbation π from (3.1) - (3.3) and obtain the system

$$\begin{aligned} \mathcal{L}w &= -\,2RML_1\theta, \\ \sigma\theta &= -\,Rw + \Delta\theta, \end{aligned} \tag{3.10}$$

where $\mathcal{L}$ and L_1 are the Tyvand & Storesletten (1991) operators given by

$$\mathcal{L} = \xi\frac{\partial^2}{\partial y^2} + M_{11}\frac{\partial^2}{\partial z^2} + M_{33}\frac{\partial^2}{\partial x^2} - 2M_{13}\frac{\partial^2}{\partial x\partial z}\,, \tag{3.11}$$

$$\begin{aligned} &= \xi\frac{\partial^2}{\partial y^2} + (\xi\cos^2\beta + \sin^2\beta)\frac{\partial^2}{\partial z^2} + (\xi\sin^2\beta + \cos^2\beta)\frac{\partial^2}{\partial x^2} \\ &\quad - 2(\xi - 1)\sin\beta\cos\beta\frac{\partial^2}{\partial x\partial z}\,, \end{aligned} \tag{3.12}$$

$$L_1 = M_{11}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\,. \tag{3.13}$$

A single equation for θ may then be obtained from (3.10) to yield

$$\mathcal{L}\Delta\theta - \sigma\mathcal{L}\theta = -2R^2ML_1\theta. \tag{3.14}$$

If we multiply this equation by θ^* and integrate over V the troublesome term is

$$-2M_{13} < \theta^*\partial_{xz}\partial_{zz}\theta >$$

where $\partial_x = \partial/\partial x$, etc. This expression may be acted upon by integration by parts to yield

$$-2M_{13} < \theta^*\partial_{xz}\partial_{zz}\theta > = -M_{13}[< \theta_{zx}\theta^*_{zz} > + < \theta_{zz}\theta^*_{xz} >] + M_{13}\oint_{z=0,1}\theta_{zx}\theta^*_z\,dx\,dy\,,$$

where we briefly use subscripts to denote differentiation, e.g. $\theta_{zx} = \partial^2\theta/\partial z\partial x$, and where the boundary integral denotes integration over those parts of ∂V lying in $z = 0, 1$. We observe that the boundary terms do *not* vanish. In fact, taking the imaginary part of the equation which results from multiplying (3.14) by θ^* and integrating over V we obtain

$$\begin{aligned} \sigma_1\Big[M_{11}\|\theta_z\|^2 + M_{33}\|\theta_x\|^2 - M_{13}(< \theta_x\theta^*_z > + < \theta_z\theta^*_x >) \\ + \xi\|\theta_y\|^2\Big] = -\mathrm{Im}\left(\oint_{z=0,1}\theta_{xz}\theta^*_z\,dx\,dy\right). \end{aligned} \tag{3.15}$$

If we introduce normal modes

$$\theta = \Theta(z)e^{i(kx+my)}$$

then from (3.15) we may derive

$$\sigma_1 = \frac{-kM_{13}\big(|\Theta_z(1)|^2 - |\Theta_z(0)|^2\big)}{\Big[M_{11}\|\Theta_z\|^2 + (M_{33}k^2 + \xi m^2)\|\Theta\|^2 + ikM_{13}(< \Theta_z\Theta^* > - < \Theta\Theta_z^* >)\Big]}\ . \tag{3.16}$$

The right hand side of this expression is real and its presence precludes any proof that $\sigma_1 = 0$; indeed, we find numerically that $\sigma_1 \neq 0$, i.e. $\sigma \in \mathbf{C}$ at the onset of convection. It is worth noting that the above problem is still present if one tries to apply the technique to the Tyvand - Storesletten system; we believe in that case a method like that outlined earlier is necessary to show exchange of stabilities.

To solve the eigenvalue problem (3.10) numerically we introduce normal modes

$$w = W(z)e^{i(kx+my)}$$

$$\theta = \Theta(z)e^{i(kx+my)}$$

and then have to solve the system

$$D^2W = 2ik\frac{M_{13}}{M_{11}}\,DW + \Big(\frac{M_{33}k^2 + \xi m^2}{M_{11}}\Big)W + 2RM(z)\Big(m^2 + \frac{k^2}{M_{11}}\Big)\Theta, \tag{3.17}$$

$$D^2\Theta = (a^2 + \sigma)\Theta + RW, \tag{3.18}$$

where $a^2 = k^2 + m^2$, and $D = d/dz$.

We have solved the above system by two entirely different numerical methods, the compound matrix method described lucidly in Drazin & Reid (1981), and the Chebyshev tau method, cf. Gardner *et al.* (1989) together with the QZ algorithm of Moler & Stewart (1973). We work directly with the system (3.17), (3.18) rather than converting to a fourth order equation and so the calculation of the eigenvalues and eigenfunctions is different from that described elsewhere. It is not the goal of this paper to go into complete detail with the numerical methods, however, and so we refer the interested reader to Straughan & Walker (1994) for further details.

A very brief sketch of the numerical techniques is included for completeness. The compound matrix method begins by defining

$$\begin{aligned}
y_1 =& W_1 W_2' - W_2 W_1',\\
y_2 =& W_1 \Theta_2 - W_2 \Theta_1,\\
y_3 =& W_1 \Theta_2' - W_2 \Theta_1',\\
y_4 =& W_1' \Theta_2 - W_2' \Theta_1,\\
y_5 =& W_1' \Theta_2' - W_2' \Theta_1',\\
y_6 =& \Theta_1 \Theta_2' - \Theta_2 \Theta_1',
\end{aligned}$$

1 and 2 denoting independent solutions obtained by setting

$$W'(0) = 1 \qquad \text{and} \qquad \Theta'(0) = 1$$

in turn. This method removes the problem of round off error associated with the standard shooting method. The initial condition for the compound matrix variables is

$$y_5(0) = 1$$

and the final condition is

$$y_2(1) = 0.$$

The y's satisfy the differential equations

$$\begin{aligned}
y_1' =& i\alpha_1 y_1 + \gamma y_2,\\
y_2' =& y_3 + y_4,\\
y_3' =& y_5 + (a^2 + \sigma) y_2,\\
y_4' =& y_5 + i\alpha_1 y_4 + \beta_1 y_2,\\
y_5' =& -Ry_1 + (a^2 + \sigma) y_4 + \beta_1 y_3 + i\alpha_1 y_5 + \gamma y_6,\\
y_6' =& -Ry_2,
\end{aligned} \tag{3.19}$$

where

$$\alpha_1 = \frac{2kM_{13}}{M_{11}},$$

$$\beta_1 = \frac{M_{33}k^2 + \xi m^2}{M_{11}},$$

and

$$\gamma = 2M(z)\left(m^2 + \frac{k^2}{M_{11}}\right).$$

System (3.19) is a system of 12 real equations and is integrated to find $R_L = R_{crit}$ and the corresponding y's.

The eigenfunctions are found by integrating the equation

$$u_1 W'' - u_2 W' + u_4 W = 0$$

from 1 to 0, where the u's are given in terms of the y's by

$$u_1 = y_1,$$

$$u_2 = \alpha_1 y_1 + \gamma R_L y_2,$$

$$u_4 = -\beta_1 y_1 + \gamma R_L y_4.$$

A similar procedure may be applied to find θ directly.

For the Chebyshev tau method the solutions W, Θ are expanded in a series of Chebyshev polynomials,

$$W = \sum_{n=0}^{N} w_n T_n(z),$$

$$\Theta = \sum_{n=0}^{N} \theta_n T_n(z),$$

and the boundary condition rows are removed in the D^2 operator, the second order differentiation operator which results from $\partial^2/\partial z^2$ when the Chebyshev representation is employed. This procedure results in having to solve the matrix system

$$\begin{pmatrix} D^2 - \beta_1 I & -2R(m^2 + k^2/M_{11})(\hat{\xi} I - N) \\ -RI & D^2 - a^2 I \end{pmatrix} \begin{pmatrix} W \\ \Theta \end{pmatrix} - \begin{pmatrix} i\alpha_1 D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} W \\ \Theta \end{pmatrix} = \sigma \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} W \\ \Theta \end{pmatrix}$$

where W and Θ denote the vectors $(w_0, \dots, w_N)$, $(\theta_0, \dots, \theta_N)$, respectively, and where N is the Chebyshev matrix which arises from the representation of the function z. The matrix system is solved for the eigenvalues σ_j and the corresponding eigenvectors $\{w_n, \theta_n\}$ from which the appropriate eigenfunctions may be found. Excellent agreement is found between the techniques.

We point out that our methods are entirely different from the one employed by Tyvand & Storesletten (1991). Also, we find the critical Rayleigh number of linear theory, R_L^2 by minimizing the expression

$$R_L^2(k, m)$$

using a quasi Newton technique, rather than by examination as was apparently done in Tyvand & Storesletten (1991). However, we do corroborate the findings of Tyvand & Storesletten (1991) and in particular we find the critical Rayleigh number with $m = 0$ if $\xi < 1$, whereas $k = 0$ if $\xi > 1$, exactly as was found in Tyvand & Storesletten (1991).

A presentation of our numerical results is deferred until section 7 after the nonlinear theory is examined.

4. A global nonlinear stability threshold by a weighted energy method

If one attempts to derive a fully nonlinear stability result from (2.11) - (2.13) by the usual method of energy, i.e. multiply (2.11) by u_i, (2.13) by θ and integrate each over V, then one faces the problem of dealing with the cubic nonlinearity $< \theta^2 w >$ which arises from (2.11). To remove this term and thereby achieve a global nonlinear stability result we resort to a weighted energy technique suggested by Payne & Straughan (1987).

First, we multiply (2.11) by u_i and integrate over V to obtain

$$0 = - < M_{ij} u_i u_j > -2R < M\theta w > + < \theta^2 w > . \tag{4.1}$$

Then, for $\mu (> 2)$ a constant to be chosen, we introduce the weight function $\hat{\mu}$ by

$$\hat{\mu} = \mu - 2z,$$

and put

$$E(t) = \frac{1}{2} < \hat{\mu} \theta^2 > . \tag{4.2}$$

We differentiate E and substitute from (2.13), and use the boundary conditions to find

$$\frac{dE}{dt} = - < w\theta^2 > -R < \hat{\mu}\theta w > - < \hat{\mu} |\nabla \theta|^2 > . \tag{4.3}$$

The effect of the weight is to keep the $u_i \theta_{,i}$ term in to yield a cubic term in (4.3). Hence, from (4.1) and (4.3) we derive

$$\frac{dE}{dt} = RI - D, \tag{4.4}$$

where

$$I = - < (2M + \hat{\mu})\theta w >,$$

$$D = < M_{ij} u_i u_j > + < \hat{\mu} |\nabla \theta|^2 > .$$

Next, define R_W by

$$\frac{1}{R_W} = \max_{\mathcal{H}} \frac{I}{D} , \tag{4.5}$$

where $\mathcal{H}$ is the space of admissible solutions. If $R < R_W$ then from (4.4) it is easy to show, cf. Straughan (1992),

$$\frac{dE}{dt} \leq -D \left(\frac{R_W - R}{R_W} \right) . \tag{4.6}$$

To proceed from this we know M_{ij} is symmetric and further may be rearranged as

$$M_{ij} u_i u_j = v^2 + \xi (u \cos\beta + w \sin\beta)^2 + (u \sin\beta - w \cos\beta)^2,$$

where $(u,v,w)=\mathbf{u}$. If $0<\xi<1$, then we rewrite this as

$$\begin{aligned}M_{ij}u_iu_j =&v^2+\big[\xi+(1-\xi)\sin^2\beta\big]u^2\\&+\big[\xi+(1-\xi)\cos^2\beta\big]w^2+2(\xi-1)\sin\beta\cos\beta\,uw\,,\\ \geq& v^2+\xi(u^2+w^2),\end{aligned}\tag{4.7}$$

and so $M_{ij}u_iu_j$ is clearly positive definite. If $\xi\geq 1$ then we arrive at the same conclusion from the rearrangement

$$\begin{aligned}M_{ij}u_iu_j =&v^2+\big[(\xi-1)\cos^2\beta+1\big]u^2\\&+\big[(\xi-1)\sin^2\beta+1\big]w^2+2(\xi-1)\sin\beta\cos\beta\,uw\,.\end{aligned}$$

Thus, since $\mu>2$ we may use Poincaré's inequality to deduce the existence of a constant $\delta>0$ such that

$$D\geq\delta E,$$

and so from (4.6), provided $R<R_W$, we conclude that $E\to 0$ at least exponentially, as $t\to\infty$. Thus, R_W represents a threshold for global (i.e. for all initial data) nonlinear stability.

Since $E\to 0$ and $\mu>2$, it also follows that $\|\theta\|\to 0$.

To calculate R_W we observe that the Euler Lagrange equations from (4.5) are

$$-R_WF\theta k_i-M_{ij}u_j=\pi_{,i}\,,\tag{4.8}$$

$$u_{i,i}=0,\tag{4.9}$$

$$\hat{\mu}\Delta\theta-2\theta_{,z}-R_WFw=0,\tag{4.10}$$

where π is a Lagrange multiplier and

$$F(z)=M(z)+\frac{1}{2}\hat{\mu}(z).$$

We may eliminate the term π from (4.8) to find

$$\mathcal{L}w=-R_WFL_1\theta.\tag{4.11}$$

Upon introducing normal modes in (4.9) - (4.11) we derive the eigenvalue equations

$$D^2W=i\alpha_1DW+\beta_1W+R_W\gamma\Theta,\tag{4.12}$$

$$D^2\Theta=\frac{2}{\hat{\mu}}D\Theta+a^2\Theta+\frac{R_W\gamma}{\hat{\mu}}W,\tag{4.13}$$

where α_1, β_1 are defined after (3.19) and γ is now given by

$$\gamma = F(z)\left(m^2 + \frac{k^2}{M_{11}}\right).$$

We may again introduce analogous compound matrix variables $y_1, \dots, y_6$, as in section 3 and from (4.12), (4.13) these may be shown to satisfy the system of equations

$$\begin{aligned}
y_1' &= i\alpha_1 y_1 + R_W \gamma y_2,\\
y_2' &= y_3 + y_4,\\
y_3' &= y_5 + \frac{2}{\hat{\mu}} y_3 + a^2 y_2,\\
y_4' &= y_5 + i\alpha_1 y_4 + \beta_1 y_2,\\
y_5' &= -\frac{R_W F}{\hat{\mu}} y_1 + \beta_1 y_3 + a^2 y_4 + \left(\frac{2}{\hat{\mu}} + i\alpha_1\right) y_5 + R_W \gamma y_6,\\
y_6' &= \frac{2}{\hat{\mu}} y_6 - \frac{R_W F}{\hat{\mu}} y_2.
\end{aligned} \tag{4.14}$$

This system is solved subject to the initial conditions $y_5(0) = 1$ and R_W is determined by requiring $y_2(1) = 0$, recollecting $y_i \in \mathbf{C}$, i.e. (4.14) is really a system of 12 real equations.

The numerical results of this calculation are reported in section 7. However, we do remark that they are very close to those obtained in section 3 for the onset of instability, and hence are very useful. In the computations we treat R_W as complex, but we find the imaginary part of R_W to be zero after computation. The numerical routine calculates R_W for fixed k, m and μ and then minimizes in k or m and afterward maximizes in μ. While we did use a quasi - Newton technique to minimize over k and m in the numerical scheme for the linear instability problem, for system (4.14) we simply checked numerically that the minimum was achieved with $m = 0$ when $\xi < 1$ and with $k = 0$ when $\xi > 1$.

5. An energy method appropriate to boundary conditions for patterned ground formation

The weighted energy technique of the last section works fine for the temperature boundary condition $\theta = 0$ on $z = 0, 1$. However, in the theory for the formation of stones into regular patterns to which the present model is eminently applicable, a boundary condition which more faithfully reflects solar radiation is

$$\frac{\partial \theta}{\partial z} = 0 \qquad \text{on} \qquad z = 1, \tag{5.1}$$

cf. George *et al.* (1989).

If we impose boundary condition (5.1) then the method of section 4 breaks down. Instead, we can develop a method which utilizes a Sobolev inequality on the cubic term, cf. Straughan (1985) for the case of penetrative convection in a fluid layer. However, for the case of a porous layer the absence of the inertia term in the Darcy equation means we have to use an "energy" which involves more than simply L^2 terms.

We commence by multiplying (2.11) by u_i, (2.13) by θ and integrating over V. Let $\lambda(>0)$ be a constant at our disposal. Then, we derive

$$\frac{d}{dt}\frac{1}{2}\lambda\|\theta\|^2 = RI - D + N, \tag{5.2}$$

where

$$I = - < (2M+\lambda)\theta w >, \tag{5.3}$$

$$D = \lambda\|\nabla\theta\|^2 + < M_{ij}u_iu_j >, \tag{5.4}$$

and

$$N =< \theta^2 w > . \tag{5.5}$$

We wish to use the variational theory of energy stability and so put

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D}. \tag{5.6}$$

Then from (5.2) we may see that

$$\frac{d}{dt}\frac{1}{2}\lambda\|\theta\|^2 \le -aD + N,$$

where we have assumed $R < R_E$, and where

$$a = \frac{R_E - R}{R_E}.$$

To handle N we may use the Cauchy - Schwarz inequality and the Sobolev inequality

$$\|\theta^2\| \le c_1\|\nabla\theta\|^2$$

to find

$$\frac{d}{dt}\frac{1}{2}\lambda\|\theta\|^2 \le -aD + c_1\|w\|\|\nabla\theta\|^2. \tag{5.7}$$

At this stage we do not see how to proceed with the standard energy. To surmount this difficulty we additionally work with the L^4 integral of θ and so let $\phi = \theta^2$, then from (2.13) we find for $\zeta > 0$ another constant we may select judiciously,

$$\frac{d}{dt}\frac{\zeta}{4}\|\phi\|^2 = -R\zeta < w\theta\phi > - \frac{3\zeta}{4}\|\nabla\phi\|^2 . \tag{5.8}$$

The idea is to define a generalised energy $\mathcal{E}(t)$ by

$$\mathcal{E}(t) = \frac{1}{2}\lambda\|\theta\|^2 + \frac{1}{4}\zeta\|\phi\|^2 . \tag{5.9}$$

From (2.11) we have

$$< M_{ij}u_iu_j >= -2R < M\theta w > + < \theta^2 w > .$$

Note that $M \leq \hat{\xi}$, and furthermore, we know M_{ij} is positive definite. In the following we suppose $0 < \xi < 1$ and so we have

$$< M_{ij}u_iu_j > \geq \xi\|\mathbf{u}\|^2,$$

but it is easy to change to $\xi \geq 1$. Thus we obtain from the last equation, the lower bound on $< M_{ij}u_iu_j >$, and the Cauchy - Schwarz inequality,

$$\xi\|\mathbf{u}\|^2 \leq 2\hat{\xi}R\|\theta\|\|w\| + \|\theta^2\|\|w\|.$$

From this inequality and definition (5.9) we may see that

$$\|\mathbf{u}\| \leq k\mathcal{E}^{1/2}, \tag{5.10}$$

where

$$k = \frac{2^{3/2}R\hat{\xi}}{\xi\lambda^{1/2}} + \frac{2}{\xi\zeta^{1/2}} .$$

Upon using (5.10) in (5.7) and (5.8) we find

$$\frac{d}{dt}\frac{1}{2}\lambda\|\theta\|^2 \leq -aD + c_1k\mathcal{E}^{1/2}D, \tag{5.11}$$

$$\frac{d}{dt}\frac{\zeta}{4}\|\phi\|^2 \leq -\frac{3\zeta}{4}\|\nabla\phi\|^2 + \frac{R\zeta kc_1}{\lambda^{1/2}}\,\mathcal{E}^{1/2}D^{1/2}\|\nabla\phi\|. \tag{5.12}$$

Next, put

$$\mathcal{D} = aD + \frac{3\zeta}{4}\|\nabla\phi\|^2$$

and add (5.11), (5.12) to obtain

$$\frac{d\mathcal{E}}{dt} \leq -\mathcal{D}(1 - A\mathcal{E}^{1/2}), \tag{5.13}$$

where

$$A = \frac{c_1k}{a}\left(1 + \frac{2R\zeta^{1/2}}{\sqrt{3\lambda}}\right) . \tag{5.14}$$

From (5.13), it is now straightforward to show that, cf. Straughan (1992), if

$$R < R_E \qquad \text{and} \qquad \mathcal{E}^{1/2}(0) < A^{-1},$$

then

$$\mathcal{E}(t) \to 0 \qquad \text{as} \qquad t \to \infty,$$

and nonlinear stability follows. Furthermore, we have decay of $\|\theta\|$, $\|\theta\|_{L^4}$ and from (5.10), $\|\mathbf{u}\|$. However, we point out that the stability obtained here is not as strong as the global stability of section 4, since it is conditional, i.e. the size of the initial energy is restricted.

The maximum problem of (5.6) is solved by putting $\psi = \sqrt{\lambda}\theta$ and then resolving

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{-<fw\psi>}{<M_{ij}u_iu_j> + \|\nabla\psi\|^2},$$

where

$$f = \frac{1}{2}\sqrt{\lambda} + \frac{M(z)}{\sqrt{\lambda}}.$$

The Euler Lagrange equations for this maximum are

$$\begin{aligned} R_E f\psi k_i + M_{ij}u_j &= \pi_{,i}\,, \\ \Delta\psi - R_E f w &= 0, \end{aligned}$$

with u_i again being divergence free. The Lagrange multiplier, π, may now be eliminated and then we must solve

$$\begin{aligned} \mathcal{L}w &= -R_E f(z) L_1\psi, \\ \Delta\psi &= R_E f(z) w. \end{aligned} \tag{5.15}$$

Upon introducing normal modes these equations reduce to

$$\begin{aligned} D^2W &= i\alpha_1 DW + \beta_1 W + R_E\gamma\Psi, \\ D^2\Psi &= a^2\Psi + R_E fW, \end{aligned} \tag{5.16}$$

where α_1, β_1 are as in (4.12), (4.13), and in the present case

$$\gamma = f(z)\left(m^2 + \frac{k^2}{M_{11}}\right).$$

The compound matrix equations for this system are

$$\begin{aligned} y_1' &= i\alpha_1 y_1 + R_E\gamma y_2, \\ y_2' &= y_3 + y_4, \\ y_3' &= y_5 + a^2 y_2, \\ y_4' &= y_5 + i\alpha_1 y_4 + \beta_1 y_2, \\ y_5' &= -R_E f y_1 + \beta_1 y_3 + a^2 y_4 + i\alpha_1 y_5 + R_E\gamma y_6, \\ y_6' &= -R_E f y_2. \end{aligned}$$

This system is solved and we find

$$R_E^2(crit) = \max_{\lambda>0} \min_{a^2} R_E(a^2, \lambda).$$

We checked that R_E has a minimum when $m = 0$ ($\xi < 1$) and $k = 0$ ($\xi > 1$) and so the minimum of $a^2 = k^2$ when $\xi < 1$ is sufficient.

The values of R_E found are extremely close to R_L. We have always found

$$R_L > R_E > R_W ,$$

although the differences are minimal.

Remark. We could develop a conditional nonlinear energy stability theory for the system (2.11) - (2.13) when the viscosity μ is a very general function of temperature, provided we either include a Brinkman viscosity term in (2.11) as was done in Richardson & Straughan (1993), or if we have present a Forchheimer term in the same equation. In the latter case we would apply the interesting device introduced by Qin *et al.* (1994). While both the papers of Richardson & Straughan (1993) and Qin *et al.* (1994) adopted only a viscosity linear in T, the analysis is extendable to a general form by utilizing an idea of Professor S. Rionero of the University of Naples which employs the intermediate value theorem. Details of an idea equivalent to the latter, applied to convection in a fluid, may be found in Straughan (1994).

6. A penetrative convection model involving an internal heat source

The heat source models of Roberts (1967) and Matthews (1988) are capable of describing penetrative convection in a fluid layer. We here show how one may develop such a model for penetrative convection in an anisotropic porous layer.

Suppose now the density relation is linear in the temperature field, i.e.

$$\rho(T) = \rho_0\left[1 - \alpha(T - T_0)\right] ,$$

ρ_0, T_0 constants, and let Q (constant) be a heat source or sink. We could allow Q to depend on z and proceed as below, *mutatis mutandis*.

The equations which replace (2.7) - (2.9) are

$$\nabla p = -\mu(\mathbf{K}^*)^{-1}\mathbf{v} + g\rho_0\alpha T\mathbf{k}, \tag{6.1}$$

$$\mathrm{div}\mathbf{v} = 0, \tag{6.2}$$

$$T_{,t} + (\mathbf{v}.\nabla)T = \kappa\Delta T + Q. \tag{6.3}$$

The basic solution whose stability we study is now

$$\bar{v}_i = 0, \qquad \bar{T} = -\frac{Q}{2\kappa}z^2 + \left[\frac{(T_1 - T_0)}{d} + \frac{Qd}{2\kappa}\right]z + T_0,$$

where T_1, T_0 now denote the temperatures of the planes $z = d$, $z = 0$, respectively. We treat explicitly the case $A = -Q(> 0)$, with T_0, T_1 prescribed and this may then be arranged to yield a layer in which the upper half wishes to be gravitationally stable whereas the lower half will tend to be unstable. Our analysis will certainly encompass other situations, but the above mimics the penetrative convection problem of the earlier sections.

By introducing perturbations and non - dimensionalizing the perturbation equations with the scales

$$\begin{aligned} &t^* = d^2/\kappa, \qquad U = \kappa/d, \qquad \mathbf{x} = \mathbf{x}^* d, \\ &(T^\sharp)^2 = \frac{Ad\mu}{\rho_0 \alpha g K_\perp}, \qquad P = \frac{d\mu U}{K_\perp}, \qquad R^2 = \frac{d^3 K_\perp \rho_0 \alpha g A}{\mu \kappa^2}, \end{aligned}$$

where R^2 is here the Rayleigh number, we derive

$$\begin{aligned} &\pi_{,i} = -M_{ij}u_j + Rk_i\theta, \\ &u_{i,i} = 0, \\ &\theta_{,t} + u_i\theta_{,i} = Rf(z)w + \Delta\theta, \end{aligned} \tag{6.4}$$

where $f(z) = \frac{1}{2} - \epsilon - z$, u_i is again solenoidal and the parameter ϵ is given by

$$\epsilon = \frac{\Delta T\,\kappa}{d^2 A}, \qquad \Delta T = T_1 - T_0. \tag{6.5}$$

We can easily develop a nonlinear energy stability theory for this model; however, since we believe the model of section 2 is more applicable to problems involving water near 4°C, we simply develop a linear analysis. It is worth remarking, however, that the nonlinear energy stability analysis for (6.4) is more straightforward than that of sections 4 and 5 and yields directly unconditional nonlinear stability. The linearised theory of instability basically reduces to solving the system of equations

$$\begin{aligned} \mathcal{L}w &= RL_1\theta, \\ \sigma\theta &= \Delta\theta + Rfw, \end{aligned} \tag{6.6}$$

where $\mathcal{L}, L_1$ are again the Tyvand - Storesletten operators. What we again find is that $\sigma \in \mathbf{C}$, i.e. the inhomogeneity introduced by the extra effect, in this case the heat source, leads to oscillatory convection being the dominant mode of instability.

7. Numerical results and conclusions

In this section Ra_L is the critical Rayleigh number for linear instability as outlined in section 3 and Ra_W similarly denotes the global nonlinear stability threshold obtained as indicated in section 4. The corresponding critical wavenumbers will likewise be denoted by subscript L or W.

We concentrate on the case $0 < \xi < 1$ because this is where the significant changes are with non-penetrative convection. When $\xi > 1$ we find that Ra_L is achieved when $k = 0$ and thus (3.17), (3.18) are essentially real equations, although in our numerical calculations we allow m and $k \neq 0$ and minimize over both variables. In the case $\xi > 1$ we effectively recover the results of Tyvand & Storesletten (1991), except for the penetration effect which becomes stronger as the temperature of the upper plate, T_1 is increased beyond 4°C. In view of this we report only one numerical result for this parameter range. It is to be observed that for $\xi > 1$ we find $\sigma \in \mathbf{R}$ at criticality.

$$\xi = 2, \quad \beta = 30, \quad T_1 = 4;$$
$$Ra_L = 41.2463, \quad k_L = 0, \quad m_L = 3.09029, \quad \sigma_1 = 0.$$

The situation $0 < \xi < 1$ means $K_{\|} > K_{\perp}$, i.e. the fluid flows easiest along the direction of the major principal axis. This is the situation we expect to be of most interest physically and is discussed below. When $0 < \xi < 1$ we always find $\sigma_1 \neq 0$ at criticality, i.e. at the values of k, m which minimize Ra, interpolating on σ_r at each stage to find where $\sigma_r = 0$. We have computed several critical values for variations of β, ξ, and T_1. To illustrate how σ_1 changes as soon as anisotropy is present in penetrative convection we first observe that with

$$\xi = 0.99, \quad \beta = 89, \quad T_1 = 4;$$
$$Ra_L = 38.3469, \quad k_L = 3.20320, \quad m_L = 0, \quad \sigma_1 = 3.02941 \times 10^{-4}.$$

This situation is very close to isotropy, and since $T_1 = 4$, the whole layer will be unstable, and there really is no penetration effect. However, even in this case, the non - Boussinesq density leads clearly to $\sigma_1 \neq 0$. For comparison purposes, we observe the critical values with a penetration effect and strong permeability difference between $K_{\|}$ and $K_{\perp}$: when

$$\xi = 0.25, \quad \beta = 30, \quad T_1 = 6;$$
$$Ra_L = 52.6846, \quad k_L = 3.13816, \quad m_L = 0, \quad \sigma_1 = -2.82059.$$

Tables 1 - 3 display critical values for variations, respectively, in β, ξ, and T_1, keeping the other two parameters fixed. The critical value of m was always zero and hence is not exhibited.

β	Ra_L	k_L	σ_1
10	27.1891	2.07790	-1.10678
20	25.7637	2.41687	-1.98831
30	25.1938	2.92345	-2.61304
40	25.4635	3.51101	-2.98425
50	26.1485	4.11561	-3.05914
60	26.9103	4.68082	-2.78100
70	27.5479	5.14690	-2.12936
80	27.9592	5.45570	-1.15762

Table 1. Critical parameters of linear instability theory, with $\xi = 0.125$, $T_1 = 5°$C.

The values of Ra_W, k_W, and coupling parameter μ, corresponding to $\beta = 40$ in the above table are

$$Ra_W = 23.07996, \qquad k_W = 3.36796, \qquad \mu = 2.037849.$$

ξ	Ra_L	Ra_W	k_L	k_W	σ_1	μ
0.1	14.8555	14.3632	3.49145	3.42142	-2.05039	2.21759
0.3	22.1901	21.3795	3.20128	3.13717	-1.03611	2.23642
0.5	27.7014	26.6762	3.15670	3.15669	-0.59181	2.23978
0.7	32.3856	31.1904	3.16099	3.09627	-0.30366	2.23989
0.9	36.5762	35.2372	3.18187	3.11645	-0.089619	2.23887

Table 2. Critical parameters of linear instability theory and nonlinear global energy stability theory. $\beta = 40°$, $T_1 = 4°$C.

T_1	Ra_L	k_L	σ_1
4	27.7014	3.1567	-0.59181
6	71.0651	3.4668	-1.59751
8	168.8964	4.6061	-3.58009
10	330.2936	5.7898	-5.64113
12	570.7258	6.9464	-8.12228

Table 3. Critical parameters of linear instability theory, with $\xi = 0.5$, $\beta = 40$.

What is immediately evident from table 1 is that Ra achieves a minimum for β between 30 and 40°, which is completely in agreement with the findings of Tyvand & Storesletten (1991). We also remark on the closeness of the linear and nonlinear critical Rayleigh numbers for $\beta = 40°$. It may be observed that σ_1 increases in magnitude as β increases from 0° and then decreases as β approaches 90°, as it should. These results corroborate the assertion of Tyvand & Storesletten (1991) by showing that in applying this theory to insulation techniques one is best with a parallel or orthogonal orientation of the fibres, since an oblique orientation always reduces the critical Rayleigh number and hence convection will begin sooner.

Table 2 shows the quantitative decrease in Ra as ξ is decreased and this is again in good agreement with the behaviour observed by Tyvand & Storesletten (1991). The increase in the magnitude of σ_1 as the ease of flow along the principal direction becomes stronger, i.e. ξ decreases, is very noticeable. However, we observe throughout the variation of ξ, the linear instability threshold is very close to the nonlinear stability value obtained. This is very important. For, when $\sigma \in \mathbf{C}$ at criticality, energy theory usually yields results which are too conservative to be of much use, cf. the rotating Bénard problem when the Prandtl number is less than one, Straughan (1992), chapter 7. In the present situation, even though the initiation of convection is via an oscillatory instability, the weighted energy method gives a threshold Ra_W below which instability cannot occur, even for large initial data, and the Ra_W values are very close to Ra_L which means Ra_W represents a useful threshold to an experimental fluid dynamicist. When we employ a nonlinear density law we do expect to find subcritical instabilities since these are predicted in the analogous problem involving a pure fluid by the work of Veronis (1963). The present writers are currently investigating numerically three - dimensional convection studies to try and determine what happens near Ra_L.

We do not include plots of streamlines as do Tyvand & Storesletten (1991). In our situation the eigenfunction is complex and it is not so obvious what is happening. We do, however, include eigenfunction plots below to show the effect of varying T_1 and how the convective motion near the bottom plate can induce fluid motion in a counter cell in the upper layer.

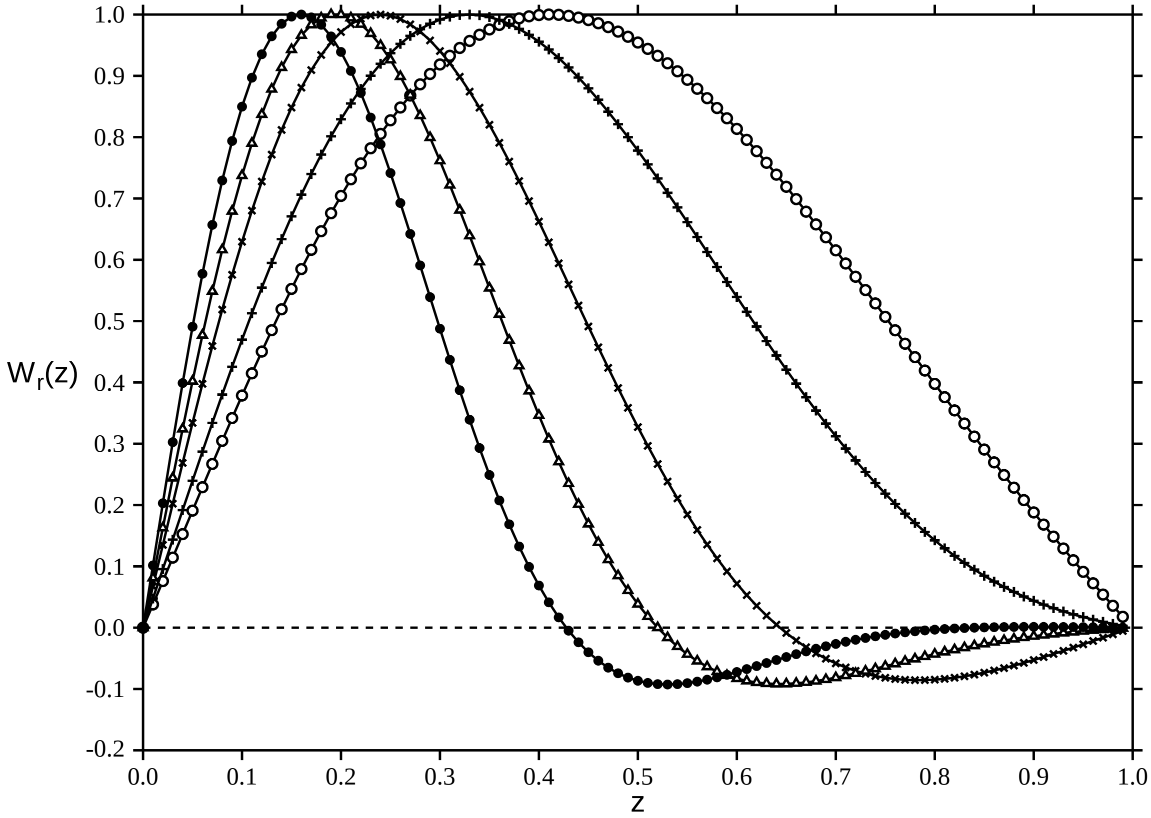

Figure 1 Plots of $W_r(z)$. Open circle ($\circ$) $T_1 = 4°$; Plus ($+$) $T_1 = 6°$; Cross ($\times$) $T_1 = 8°$; Triangle ($\triangle$) $T_1 = 10°$; Filled circle ($\bullet$) $T_1 = 12°$.

Finally, we note the strong effect the variation of T_1 has on Rayleigh number, wave number, and complex growth rate once the penetration effect is increased. In figure 1 we include plots of the real part of the eigenfunction $W_r(z)$ corresponding to table 3. One may clearly see in the plots for $T_1 = 8, 10, 12$ that the upper and lower cells have formed and increasing of T_1 leads to a stronger boundary layer effect near $z = 0$. The value of T_1 at which true penetrative convection is first seen, i.e. when W_r changes sign, we computed at

$$T_1 \approx 6.4°\mathrm{C},$$

which corresponds to

$$\lambda \approx 1.6$$

in Veronis' (1963) notation.

That the eigenfunction profiles, and hence streamline profiles, should be distorted is explained by considering the "effective force" which is experienced by a fluid "parcel" when subject to a linear density law or quadratic one. In the linear case this force looks like T whereas in the penetrative convection case it is $(T-4)^2$. Thus, consider two similar situations, (i) the linear density law with lower boundary maintained at 8°C whereas the upper is 4°C, and (ii) the quadratic law (2.1) with the lower boundary at 0°C while the upper one is kept at 4°C. Clearly, in the former, the effective force decreases linearly from 8 to 4 as we traverse the layer in the upward direction. In the second case, however, the equivalent effective force begins at 16 and decreases parabolically to *zero* in ascending the layer. The acceleration a fluid parcel experiences, therefore, is vastly different, and so we expect vastly different velocity profiles. As T_1 increases beyond 4°C the effective force changes yet again and a corresponding change in the velocity profile is to be expected. When $0 < \xi < 1$, however, the velocity profiles, i.e. u and w, are yet more complex due to the anisotropy effect.

In the heat source problem of section 6, we have computed the linear instability behaviour for various parameter ranges. However, we continue to find $\sigma_1 \neq 0$ at criticality. We present below values when the parameter ϵ defined in (6.5) is $\epsilon = 0.5$; this then yields $f(z) = 1 - z$ in (6.4) and is (in some sense) a comparison with the quadratic density model when $T_1 = 4$. The results below correspond to the onset of convection.

$$\xi = 0.25, \quad \beta = 30, \quad T_1 = 4;$$
$$Ra_L = 41.5915, \quad k_L = 3.1382, \quad m_L = 0, \quad \sigma_1 = 1.19279.$$

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References

Drazin, P.G. & Reid, W.H. 1981. *Hydrodynamic stability.* Cambridge University Press.

Gardner, D.R., Trogdon, S.A. & Douglas, R.W. 1989. A modified tau spectral method that eliminates spurious eigenvalues. *J. Computational Physics,* **80**, 137–167.

George, J.H., Gunn, R.D. & Straughan, B. 1989. Patterned ground formation and penetrative convection in porous media. *Geophys. Astrophys. Fluid Dyn.* **46**, 135–158.

Matthews, P.C. 1988. A model for the onset of penetrative convection. *J. Fluid Mech.* **188**, 571–583.

Moler, C.B. & Stewart, G.W. 1973. An algorithm for generalized matrix eigenproblems. *SIAM. J. Numerical Anal.* **10**, 241–256.

Nield, D.A. & Bejan, A. 1992. *Convection in porous media.* Springer-Verlag, Berlin-Heidelberg-New York.

Or, A.C. 1989. The effects of temperature-dependent viscosity and the instabilities in the convection rolls of a layer of fluid-saturated porous medium. *J. Fluid Mech.* **206**, 497–515.

Payne, L.E., Song, J.C. & Straughan, B. 1988. Double diffusive porous penetrative convection; thawing subsea permafrost. *Int. J. Engng. Sci.* **26**, 797–809.

Payne, L.E. & Straughan, B. 1987. Unconditional nonlinear stability in penetrative convection. *Geophys. Astrophys. Fluid Dyn.* **39**, 57–63. (See also, Corrected and extended numerical results. *Geophys. Astrophys. Fluid Dyn.* **43**, 307–309.)

Qin, Y. & Kaloni, P.N. 1992. Steady convection in a porous medium based upon the Brinkman model. *IMA J. Appl. Math.* **35**, 85–95.

Qin, Y. & Kaloni, P.N. 1993. A nonlinear stability problem of convection in a porous vertical slab. *Physics Fluids A,* **5**, 2067–2069.

Qin, Y. & Kaloni, P.N. 1994. Convective instability in an anisotropic porous medium. *Studies in Appl. Math.* **91**, 189–204.

Qin, Y., Guo, J. & Kaloni, P.N. 1995. Double diffusive penetrative convection in porous media. *Int. J. Engng. Sci.,* **33**, 303–312.

Richardson, L.L. & Straughan, B. 1993. Convection with temperature dependent viscosity in a porous medium: nonlinear stability and the Brinkman effect. *Atti Accad. Naz. Lincei,* (Ser IX) **4**, 223–230.

Roberts, P.H. 1967. Convection in horizontal layers with internal heat generation. Theory.

J. Fluid Mech. **30**, 33–49.

Straughan, B. 1985. Finite amplitude instability thresholds in penetrative convection. *Geophys. Astrophys. Fluid Dyn.* **34**, 227–242.

Straughan, B. 1992. *The energy method, stability, and nonlinear convection.* Springer-Verlag, Berlin-Heidelberg-New York.

Straughan, B. 1993. *Mathematical aspects of penetrative convection.* Longman, Harlow.

Straughan, B. 1994. Stability thresholds for convection when the viscosity has a general form of temperature dependence. In, *Energy Methods in Continuum Mechanics,* ed. S.N. Antontsev, J.I. Diaz, and S.I. Shmarev, Kluwer Academic, in the press.

Straughan, B. & Walker, D.W. 1994. Two very accurate and efficient methods for computing eigenvalues and eigenfunctions in porous convection problems. To appear.

Tyvand, P.A. & Storesletten, L. 1991. Onset of convection in an anisotropic porous medium with oblique principal axes. *J. Fluid Mech.* **226**, 371–382.

Veronis, G. 1963. Penetrative convection. *Astrophys. J.* **137**, 641–663.