Euler's Formula: Phasor Form

Euler's Formula[3] states that we can express the trigonometric form as:

$$\mathbf{e}^{\mathbf{i}\phi} = \cos\phi + \mathbf{i}\sin\phi, \quad \phi \in \mathbb{R}$$

Exercise: Show that

$$\mathbf{e}^{-\mathbf{i}\phi} = \cos\phi - \mathbf{i}\sin\phi$$

This is also known as **phasor form** or a **Phasor**, for short.

Note: Phasors and the related trigonometric form are **very important** to **Fourier Theory** which we study later.

[3] we won't prove this here. Proof here if interested

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Phasor Notation

General Phasor Form: $re^{i\phi}$

More generally we use $re^{i\phi}$ where:

$$re^{i\phi} = r(\cos\phi + i\sin\phi)$$

MATLAB Speaks the Phasor Language

MATLAB Complex No. Phasor Declaration

```
>> exp( 1i*(pi/4) )

ans =   0.7071 + 0.7071i

>> [abs(z), angle(z)]

ans =   1.0000      0.7854
```

Phasors are stunning!

Phasers on stun!

Phasors are stunning!

Phasors are very useful mathematical tools

- Can simplify Trigonometric proofs, Trig. expression manipulation *etc*
  - Can do Trigonometry without Trigonometry (well almost!)
- Electrical Signals: Can apply simplify AC circuits to DC circuit theory (*e.g.* Ohm's Law)!
- Power engineering: Three phase AC power systems analysis
- **Signal Processing**: **Fourier Theory**, **Filters**

Trig. Example: sin and cos as functions of e

From **Euler's Formula** we can write:

$$\cos\phi = \frac{\mathbf{e}^{\mathbf{i}\phi} + \mathbf{e}^{-\mathbf{i}\phi}}{\mathbf{2}}$$

$$\sin\phi = \frac{\mathbf{e}^{\mathbf{i}\phi} - \mathbf{e}^{-\mathbf{i}\phi}}{\mathbf{2i}}$$

Prove the above

Trig. Exercise: Powers of the Trigonometric Form (de Moivre's Theorem)

If n is an **integer** then show that:

$$(\cos\theta + \mathbf{i}\sin\theta)^{\mathbf{n}} = \cos\mathbf{n}\theta + \mathbf{i}\sin\mathbf{n}\theta.$$

This is known as **de Moivre's Theorem**

Complex Number Multiplication in Polar Form

Let $z_1 = [r_1, \phi_1]$ and $z_2 = [r_2, \phi_2]$ then
$z_1 = r_1(\cos\phi_1 + i\sin\phi_1)$ and $z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$
Therefore:

$$
\begin{aligned}
z_1 z_2 &= [r_1(\cos\phi_1 + i\sin\phi_1)] \times [r_2(\cos\phi_2 + i\sin\phi_2)] \\
&= r_1 r_2[(\cos\phi_1 + i\sin\phi_1) \times (\cos\phi_2 + i\sin\phi_2)] \\
&= r_1 r_2[\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 \\
&\quad +i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)]
\end{aligned}
$$

From **trigonometry** we have the following relations:

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B,$
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B,$

So **finally** we have:

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2[\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)] \\
\mathbf{z_1 z_2} &= [\mathbf{r_1 r_2}, \phi_1 + \phi_2]
\end{aligned}
$$

Complex Number Multiplication via Phasors

Alternatively, we can multiply complex numbers via **Phasors**:
$z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$.
Therefore:

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\phi_1} \times r_2 e^{i\phi_2} \\ &= r_1 r_2 e^{i\phi_1} e^{i\phi_2} \end{aligned}$$

Now in general, $e^x e^y = e^{(x+y)}$

So we get: $z_1 z_2 = r_1 r_2 e^{i(\phi_1+\phi_2)}$ which (as we should expect) gives:

$$\mathbf{z_1 z_2} = [\mathbf{r_1 r_2}, \phi_1 + \phi_2]$$

This is a much **easier** way to prove this fact — **Agree?**[4]

[4] **This sort of algebra is important for Fourier Theory later**

Complex Number Division in Phasor Form

Sticking with the **Phasor** formulation, we can **divide** two complex numbers:
$z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$.
Therefore:

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1 e^{i\phi_1}}{r_2 e^{i\phi_2}} \\
&= \frac{r_1}{r_2}\frac{e^{i\phi_1}}{e^{i\phi_2}} \\
&= \frac{r_1}{r_2} e^{i\phi_1} e^{-i\phi_2}, \quad \text{by a same argument as in multiplication} \\
&= \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)} \\
\frac{\mathbf{z_1}}{\mathbf{z_2}} &= [\frac{\mathbf{r_1}}{\mathbf{r_2}}, \phi_1 - \phi_2]
\end{aligned}$$

Exercise: Prove this formula via the trigonometric polar form — **yawn**!.

Exercises: Complex Number Multiplication and Division

- If $z_1 = 3\sqrt{2} + 3\sqrt{2}i$ and $z_2 = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$, find $z_1 z_2$ and $\frac{z_1}{z_2}$, leave your answer in **polar form**.

- Evaluate $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^3$, give your answer in **Cartesian form**.

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Complex Number Multiplication: Geometric Representation

Multiplying a complex number $z = x + iy$ by i **rotates** the vector representing z through 90° **anticlockwise**

Example: Let $z_1 = 1$.

Then

$$z_2 = iz_1 = i.$$

- Polar form of $z_1 = [1, 0^\circ]$.
- Polar form of $z_2 = [1, 90^\circ]$, **Q.E.D**.

Back to Phase: Important Example

Concept: A **phasor** is a **complex number** used to represent a **sinusoid**.

In particular:

Sinusoid : $x(t) = M\cos(\omega t + \phi), \ -\infty < t < \infty$ — a function of **time**

Phasor : $X = Me^{i\phi} = M\cos(\phi) + iM\sin(\phi)$ — a **complex number**

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Complex Numbers and Phase: Important Example

Phasors and Sinusoids are related:

$$
\begin{aligned}
\Re[Xe^{i\omega t}] &= \Re[Me^{i\phi}e^{i\omega t}] \\
&= \Re[Me^{i(\omega t+\phi)}] \\
&= \Re[M(\cos(\omega t+\phi) + i\sin(\omega t+\phi))] \\
&= M\cos(\omega t+\phi) \\
&= \mathbf{x(t)}
\end{aligned}
$$

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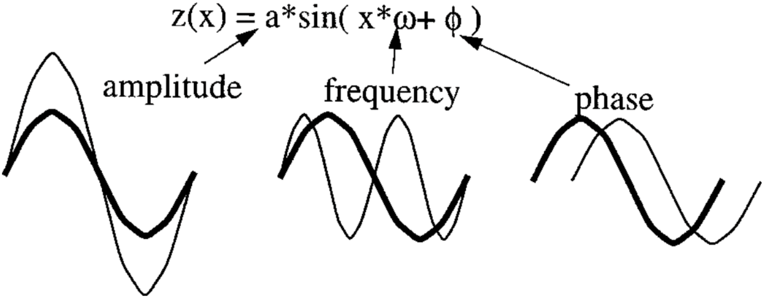
Visualising Sinusoids of differing Phase, Amplitude and Frequency



MATLAB Sine Wave Frequency and Amplitude (only)

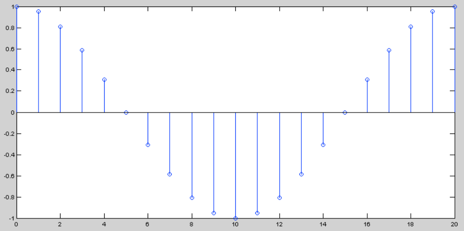
```
% Natural frequency is 2*pi radians
% If sample rate is F_s HZ then 1 HZ is 2*pi/F_s
% If wave frequency is F_w then frequency is
%       F_w* (2*pi/F_s)
% set n samples steps up to sum duration nsec*F_s where
% nsec is the duration in seconds
% So we get y = amp*sin(2*pi*n*F_w/F_s);
amp = 0.5;
F_s = 11025;
F_w = 440;
nsec = 2;
dur= nsec*F_s;
n = 0:dur;

y = amp*sin(2*pi*n*F_w/F_s);
figure(1)
plot(y(1:500));
title('N second Duration Sine Wave');
```

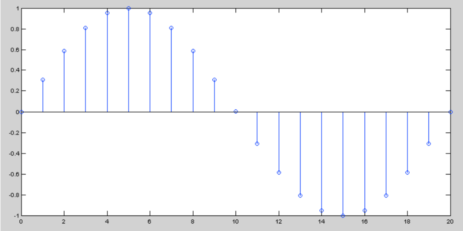
MATLAB Cos v Sin Wave

```
% Cosine is same as Sine (except 90 degrees out of phase)

yc = amp*cos(2*pi*n*F_w/F_s);

figure(2);
hold on;
plot(yc,'b');
plot(y,'r');
title('Cos (Blue)/Sin (Red) Plot (Note Phase Difference)');
hold off;
```

Sin and Cos (stem) plots

MATLAB functions `cos()` and `sin()`.





Amplitudes of a Sine Wave

Code for sinampdemo.m

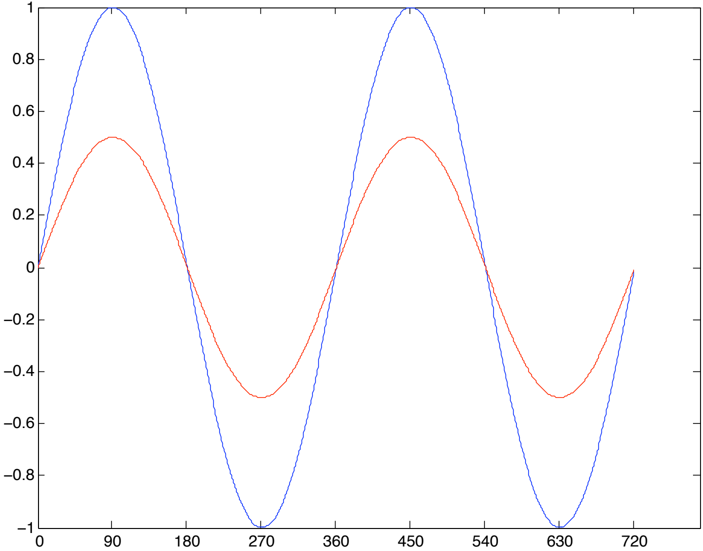
```
% Simple Sin Amplitude Demo
samp_freq = 400;
dur = 800; % 2 seconds
amp = 1; phase = 0; freq = 1;
s1 = mysin(amp,freq,phase,dur,samp_freq);

axisx = (1:dur)*360/samp_freq; % x axis in degrees
plot(axisx,s1);
set(gca,'XTick',[0:90:axisx(end)]);

fprintf('Initial Wave: \t Amplitude = ...\n', amp,
              freq, phase,...);

% change amplitude
amp = input('\nEnter Amplitude:\n\n');

s2 = mysin(amp,freq,phase,dur,samp_freq);
hold on;
plot(axisx, s2,'r');
set(gca,'XTick',[0:90:axisx(end)]);
```

mysin MATLAB code

mysin.m — a modified version of previous MATLAB sin example to account for phase

```
function s = mysin(amp,freq,phase,dur, samp_freq)
% example function to so show how amplitude,frequency
% and phase are changed in a sin function
% Inputs: amp - amplitude of the wave
%         freq - frequency of the wave
%         phase - phase of the wave in degree
%         dur - duration in number of samples
%         samp_freq - sample frequncy

x = 0:dur-1;
phase = phase*pi/180;

s = amp*sin( 2*pi*x*freq/samp_freq + phase);
```

Amplitudes of a Sine Wave: `sinampdemo` output

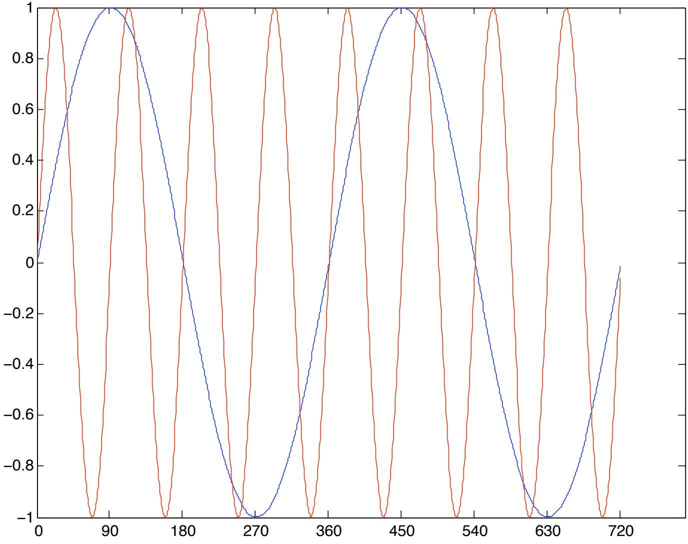


Frequencies of a Sine Wave

Code (fragment) for sinfreqdemo.m

```
% Simple Sin Frequency Demo
samp_freq = 400;
dur = 800; % 2 seconds
amp = 1; phase = 0; freq = 1;
s1 = mysin(amp,freq,phase,dur,samp_freq);

axisx = (1:dur)*360/samp_freq; % x axis in degrees
plot(axisx,s1);
set(gca,'XTick',[0:90:axisx(end)]);

fprintf('Initial Wave: \t Amplitude = %d\n', amp, freq,


% change amplitude
freq = input('\nEnter Frequency:\n\n');

s2 = mysin(amp,freq,phase,dur,samp_freq);
hold on;
plot(axisx, s2,'r');
set(gca,'XTick',[0:90:axisx(end)]);
```

Frequencies of a Sine Wave: `sinfreqdemo` output

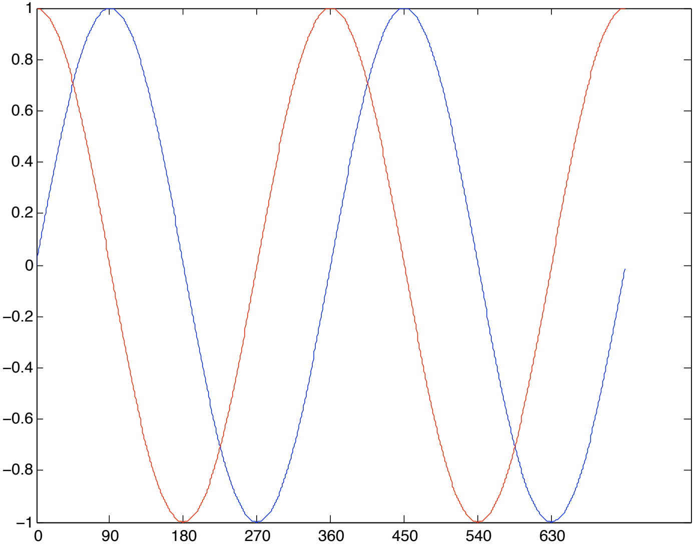


Phase of a Sine Wave

sinphasedemo.m (Fragment)

```
% Simple Sin Phase Demo
samp_freq = 400;
dur = 800; % 2 seconds
amp = 1; phase = 0; freq = 1;
s1 = mysin(amp,freq,phase,dur,samp_freq);

axisx = (1:dur)*360/samp_freq; % x axis in degrees
plot(axisx,s1);
set(gca,'XTick',[0:90:axisx(end)]);

fprintf('Initial Wave: \t Amplitude = %d\n', amp, freq,


% change amplitude
phase = input('\nEnter Phase:\n\n');

s2 = mysin(amp,freq,phase,dur,samp_freq);
hold on;
plot(axisx, s2,'r');
set(gca,'XTick',[0:90:axisx(end)]);
```

Phase of a Sine Wave: `sinphasedemo` output



Sum of Two Sinusoids of Same Frequency (1)

Hopefully we now have a good understanding and can visualise Sinusoids of different phase, amplitude and frequency.

Back to Phasors: $X = Me^{i\phi} = M\cos(\phi) + iM\sin(\phi)$

Consider two sinusoids: Same frequency, ω but different phase, θ and ϕ and amplitude, A and B

$$\mathbf{A}\cos(\omega\mathbf{t} + \theta), \text{ and}$$

$$\mathbf{B}\cos(\omega\mathbf{t} + \phi)$$

Let's add them together

Sum of Two Sinusoids of Same Frequency (2)

$$
\begin{aligned}
A\cos(\omega t+\theta)+B\cos(\omega t+\phi) &= \Re[Ae^{i(\omega t+\theta)}+Be^{i(\omega t+\phi)}] \\
&= \Re[e^{i\omega t}(Ae^{i\theta}+Be^{i\phi})]
\end{aligned}
$$

Now let $Ae^{i\theta}+Be^{i\phi}=Ce^{i\gamma}$ for some C and γ, then

$$
\begin{aligned}
\Re[e^{i\omega t}(Ae^{i\theta}+Be^{i\phi})] &= \Re[e^{i\omega t}(Ce^{i\gamma})] \\
&= C\cos(\omega t+\gamma)
\end{aligned}
$$

Sum of Two Sinusoids of Same Frequency (3)

Trigonometry Equation

$$A\cos(\omega t+\theta)+B\cos(\omega t+\phi)=C\cos(\omega t+\gamma)$$

Equivalent Complex Number Equation

$$Ae^{i\theta}+Be^{i\phi}=Ce^{i\gamma}$$

Which is neater? **Let's see**

Example: Sum of Two Sinusoids of Same Frequency (1)

Simplify

$$5\cos(\omega t + 53^\circ) + \sqrt{2}\cos(\omega t + 45^\circ)$$

Hard way via trigonometry

- Use the cosine addition formula three times
  - see **maths formula sheet handout** for formula
- Third time to simplify the result.
- Not difficult but **tedious**!

Example: Sum of Two Sinusoids of Same Frequency (2)

Easy Way Phasors

$$\begin{aligned} \Re[5e^{i53^\circ} + \sqrt{2}e^{i45^\circ}] &= (3+4i) + (1+i) \\ &= (4+5i) \\ &= 6.4e^{i51^\circ} \end{aligned}$$

So:

$$\begin{aligned} 5\cos(\omega t + 53^\circ) + \sqrt{2}\cos(\omega t + 45^\circ) &= \Re[6.4e^{i(wt+51^\circ)}] \\ &= 6.4\cos(\omega t + 51^\circ) \end{aligned}$$

This is a **very important example** - make sure you **understand it.**

Another Example (1)

Simplify

$$\cos(\omega \mathbf{t} + \mathbf{30}^\circ) + \cos(\omega \mathbf{t} + \mathbf{150}^\circ) + \sin(\omega \mathbf{t})$$

First trick to note:

$$\sin(\omega t) = \cos(\omega t - 90^\circ)$$

So now simplify:

$$\cos(\omega \mathbf{t} + \mathbf{30}^\circ) + \cos(\omega \mathbf{t} + \mathbf{150}^\circ) + \cos(\omega \mathbf{t} - \mathbf{90}^\circ)$$

Hard way via trigonometry

- Use the cosine addition formula three times
  - see **maths formula sheet handout** for formula
- Not difficult but **tedious**!

Another Example (2)

Easy Way Phasors

$$e^{i30^\circ} + e^{i150^\circ} + e^{-i90^\circ} = e^{i90}(= i)$$

So we get:

$$\Re[e^{i90}] = \cos(90^\circ) = 0$$

So:

$$\cos(\omega \mathbf{t} + \mathbf{30}^\circ) + \cos(\omega \mathbf{t} + \mathbf{150}^\circ) + \cos(\omega \mathbf{t} - \mathbf{90}^\circ) = 0$$

or

$$\cos(\omega \mathbf{t} + \mathbf{30}^\circ) + \cos(\omega \mathbf{t} + \mathbf{150}^\circ) + \sin(\omega \mathbf{t}) = 0$$

This fact is used in **three-phase AC** to conserve current flow