CM2202: Scientific Computing and Multimedia Applications
General Maths: 2. Algebra - Factorisation

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Factorisation

Factorisation is a way of simplifying **algebraic expressions**.

- As we have seen, this can be effective when finding the roots of an equation $f(x) = 0$.
- There are many other uses where factorisation can simplify the maths *e.g.* **algebraic fractions**

Simple factorisation examples:

- $\frac{25ab^2-15a^2b}{40ab^2-24a^2b} = \frac{5ab(5b-3a)}{8ab(5b-3a)} = \frac{5}{8}$
- $x^2 - 2x + 1 = (x-1)^2$
- $x^2 - 3x + 2 = (x-1)(x-2)$

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Methods and Tools for Factorisation

There are many methods and tools you can use to factorise expressions

- Common factors
- Common factors by grouping
- The *ac* **method**
- **Know your algebra**! - **Practice makes Perfect!**

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Common factors

The simplest form of factorisation:

- Extract the **highest common factors** (**HCF**) from an expression
  - These can be **variables** and/or **integers (coefficients)**.

Example:

$$24x^2y^2 - 18xy^3$$

- The HCF of the coefficients is 6
- The HCF of x is x
- The HCF of y is y^2

So

$$24x^2y^2 - 18xy^3 = 6xy^2(4x - 3y)$$

Exercise: Common factors

Factorise:

- $8x^4y^3 + 6x^3y^2 =$

- $15a^3b - 9a^2b^2 =$

Common factors by grouping

Multiple termed expressions can sometimes be **factorised** into binomial expressions by extracting common factors from each, *e.g.*:

$$\begin{aligned} 2ac + 6bc + ad + 3bd &= (2ac + 6bc) + (ad + 3bd) \\ &= 2c(a + 3b) + d(a + 3b) \\ &= (a + 3b)(2c + d) \end{aligned}$$

The trick here is to spot the factors!

Exercise: Common factors by grouping

Factorise:

- $x^3 + 4x^2y - xy - 4y^2 =$

Know your Algebra!

Recap: General Rules of Algebraic Multiplication:

- $c(a+b) = ac + bc$
- $(a+b)(c+d) = ac + ad + bc + bd$
- $(a+b)^2 = (a+b)(a+b) = aa + ba + ab + bb = a^2 + 2ab + b^2$
- $(a-b)(a+b) = aa + ab - ba - bb = a^2 + ab - ab - b^2 = a^2 - b^2$

Examples: Expanding/Simplifying Expressions

We are quite used to expanding algebraic expressions:

- $3(6a + 3b - c) - 5(2a - b + 3c) =$
  $18a + 9b - 3c - 10a + 5b - 15c = 8a + 14b - 18c$
- $2(3m - n) + 4(m + 2n) - 3(2m + 3n) =$
  $6m - 2n + 4m + 8n - 6m - 9n = 4m - 3n$
- $(x + 1)(x + 6) = x^2 + x + 6x + 6 = x^2 + 7x + 6$
- $(x + y)(m + n) = mx + nx + my + ny$
- $(x + 4)^2 = x^2 + 8x + 16$
- $(x - 3)(x + 3) = x^2 - 9$

Examples: Factorisation

Factorisations of the following expressions are:

- $8x^2 - 12x = 4x(2x - 3)$
- $5x^2 + 15x^3 = 5x^2(1 + 3x)$
- $x^2 - 2x - 15 = (x + 3)(x - 5)$
- $x^2 + 9x + 20 = (x + 5)(x + 4)$
- $xz + 2yz - 2y - x = xz - x + 2yz - 2y = x(z - 1) + 2y(z - 1) = (z - 1)(x + 2y)$
- $2x^2 - 3x - 2 = (2x + 1)(x - 2)$

The ac Method

Guidance for **factorising** quadratics of the type $ax^2 + bx + c$ where $a \neq 0$

- Obtain $|ac|$ *i.e.* the numerical value of the product ac ignoring the sign of the product.
- Write down all the possible pairs of factors of $|ac|$
  - If c is positive, we select the two factors of $|ac|$ whose sum is **equal** to $|b|$: both of these factors have the same sign as b.
  - If c is negative, we select the two factors of $|ac|$ which **differ** by the value of $|b|$; the numerically larger of these two factors has the **same** sign as that of b and the other has the **opposite** sign.
  - In each case, denote the two factors obtained as f_1 and f_2
- Then $ax^2 + bx + c$ is now written $ax^2 + \mathbf{f_1 x} + \mathbf{f_2 x} + c$ and this is **factorised** by **finding common factors**.

Example:Factorising quadratics $ax^2 + bx + c,\ a \neq 0$

Factorise:

$$6x^2 + 11x + 3$$

So we have

- $a = 6,\ b = 11,\ c = 3$
  - Therefore $|ac| = 18$
  - Factors of 18: $(1, 18), (2, 9), (3, 6)$
- c is **+ve** therefore:
  - f_1 and f_2 **should** add up to $|b| = \mathbf{11}$
  - Therefore the required factors are $(2, 9)$

We therefore write:

$$\begin{aligned} 6x^2 + 11x + 3 &= 6x^2 + 2x + 9x + 3 \\ &= (6x^2 + 9x) + (2x + 3) \\ &= 3x(2x + 3) + 1(2x + 3) \\ &= (3x + 1)(2x + 3) \end{aligned}$$

Exercise:Factorising quadratics $ax^2 + bx + c$, $a \neq 1$ or 0

Factorise:

- $3x^2 - 14x + 8 =$
- $8x^2 + 18x - 5 =$

Factors as Algebraic Fractions

Trivially: Any factors which appear in **BOTH** the **numerator** and **denominator** are called **common factors** and can simply be **cancelled**

Simple example:

$$\frac{18x^2}{6x} = 3x$$

This is a useful property to similarly apply to algebraic fractions:

- A fraction is expressed in its simplest form by **factorising** the **numerator** and **denominator** and **cancelling any common factors**.

Example: Simplify

$$\frac{x^2 - 1}{x^2 + 3x + 2} = \frac{(x-1)(x+1)}{(x+2)(x+1)} = \frac{x-1}{x+2}$$

Cubic Equations

If $f(x)$ is a cubic function $ax^3 + bx^2 + cx + d$ then

- the equation $f(x) = 0$ can have **up** to **three real roots**.

Note:

- The number of real roots will depend upon the values of a, b, c and d.
- **Factors** of the cubic clearly give us its roots. *e.g.*s:

$$x^3 - 6x^2 + 11x - 6 = 0$$

can be written as $(x-1)(x-2)(x-3) = 0$.

  - Consequently, the values of x which satisfy this equation are $x = 1$, $x = 2$ and $x = 3$

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Factor Theorem and Remainder Theorem (1)

Factor theorem

Definition: If for a given polynomial function $f(x)$,$f(a) = 0$ then $x - a$ is a factor of the polynomial $f(x)$.

Example: Factorise $2x^3 + x^2 - 13x + 6$

- **If $a = 1$ then $f(1) = 2 + 1 - 13 + 6 \neq 0$ — $(x - 1)$ is not a factor**
- **If $a = -1$ then $f(-1) = -2 + 1 + 13 + 6 \neq 0$ — $(x + 1)$ is not a factor**
- **If $a = 2$ then $f(2) = 16 + 4 - 26 + 6 = 0$ — $(x - 2)$ is a factor**

Factor Theorem and Remainder Theorem (2)

To find other factors of $2x^3 + x^2 - 13x + 6$ we can now factor out $x - 2$.

We need to do arithmetic (long) division:

$$\begin{array}{r l}
 & 2x^2 + 5x - 3 \\
x - 2 \,) & 2x^3 + x^2 - 13x + 6 \\
 & \underline{2x^3 - 4x^2} \\
 & \quad 5x^2 - 13x + 6 \\
 & \quad \underline{5x^2 - 10x} \\
 & \qquad -3x + 6 \\
 & \qquad \underline{-3x + 6} \\
 & \qquad\quad 0
\end{array}$$

Factor Theorem and Remainder Theorem (3)

It follows that $2x^2 + 5x - 3$ is also a factor of $2x^3 + x^2 - 13x + 6$.

Now

$$2x^2 + 5x - 3 = (2x - 1)(x + 3)$$

Therefore $(2x - 1)(x + 3)$ are factors too

So **all** the factors of $2x^3 + x^2 - 13x + 6$ are:

$$(x - 2)(2x - 1)(x + 3)$$

Exercise: Factor Theorem

Use the **factor theorem** to factorise the following polynomial $x^3 + 3x^2 - x - 3$:

What are the **roots** of the equation: $x^3 + 3x^2 - x - 3 = 0$?

Factor Theorem and Remainder Theorem (4)

Remainder theorem
: When $f(x)$ is divided by $(x-a)$ remainder is $f(a)$.

Example: Find the remainder when x^3+6x^2+7x-4 is divided by $x+3$

- **Using the remainder theorem** $a=-3$
- **So** $f(a)=f(x=a=3)$, **(setting** $x=-3$**)**

$$x^3+6x^2+7x-4=(-3)^3+6(-3)^2+7(-3)-4=-27+54-21-4=2$$

Exercise 1: Remainder Theorem

When the polynomial $f(x) = x^3 + 8x^2 + kx + 10$ is divided by $x - 2$, there is a remainder of 84. **Show** that $k = 17$.

Exercise 2: Remainder Theorem

Given that $x+3$ is a **factor** of $2x^3+9x^2+\mathbf{a}x-6$, **find** the value of a.

Using **this value of** a, **solve** the equation $2x^3+9x^2+\mathbf{a}x-6=0$.

Sum or difference of two cubes

Another formula for factoring is the **sum** or **difference** of **two** cubes.
The **sum** can be represented by

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

and the **difference** by

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Quartic Equations

Recap: A Polynomial of Order 4:

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

In principal, the solution to a Quartic uses the same tools as for a cubic and quadratic - it's just a little more long winded or more complex.

- The same idea, as just introduced with cubics, of using the **Factor Theorem** applies

Closed form Quartic Solution

A **closed form** solution for a quartic actually does exist:

First part enlarged:

$$r_1 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}\right)^{\frac{1}{3}}} \cdots}$$

$$r_2 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}\right)^{\frac{1}{3}}} \cdots}$$

However, in practice, this is too unwieldy to be used for solving quartic equations
(Source http://planetmath.org/encyclopedia/QuarticFormula.html)

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Matlab to the Rescue

We can use MATLAB to relieve the stress of factorisation:

```
>> syms x;
>> f = 4*x^4 - 3*x^3 - 2*x^2 + x;
>> factor(f)

ans =

x*(x - 1)*(4*x^2 + x - 1)

>> double(solve(f))

ans =

         0
    1.0000
    0.3904
   -0.6404
```