CM2202: Scientific Computing and Multimedia Applications
Linear Algebra: 2. Vectors

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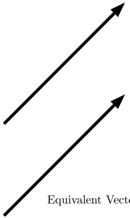
2- 3- and *n*-dimensional vectors

Vector basics

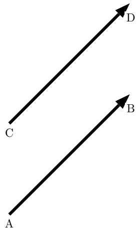
Definition (2-dimensional vectors)

We define two-dimensional vectors as directed arrows in the plane. A vector is determined by the length and the direction of the arrow. Two vectors are called equivalent if they have the same length and direction.



Equivalent Vectors

Example (2-dimensional vectors)

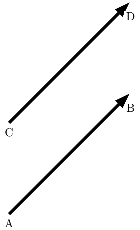


Vectors can be determined by *two points*. E.g. the vectors **AB** and **CD**.

In the above example:

- A is called the *tail* of the vector **AB**.
- B is called the *head* of the vector **AB**.

Definition (Equivalence of vectors)



Although **AB** and **CD** have different heads and tails, they are *equivalent*.

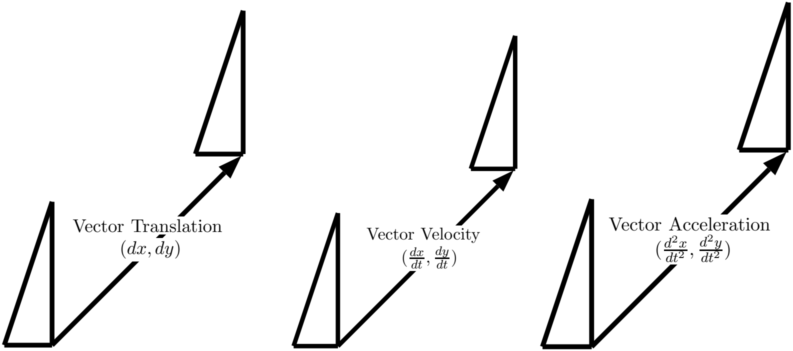
- We **distinguish** vectors **only** by their *direction* and *length*.

Thus we treat equivalent vectors as equal.
E.g.

$$\mathbf{AB} = \mathbf{CD}$$

Example (Some Real World Examples of Vectors)

Vectors can be used to represent translation (motion), velocity, acceleration:



Vector Representation/Notation

Vectors can be defined in a variety of ways:

- As we have seen already by **two points**.

  In which case we use the notation $\mathbf{AB}$.

  Alternative notations (which we do not use but you may see in some books) are $\underline{AB}$ or $\underset{\sim}{AB}$ or $\tilde{AB}$

- A vector may also be defined as a **line** whose tail is the origin and whose head coordinates are given as a (x, y) pair (and similar for higher dimensions — **more soon**.

  In this case we use the notation $\mathbf{a} = (x, y)$ or $\mathbf{a} = (x, y)$

Some Other Vector Notations You May See

An alternative notations $\langle x, y \rangle$, $\underline{a}$, $\tilde{a}$ or, even, $\underset{\sim}{a}$.

Standard Vector/Matrix Notation Conventions

Note: It is standard notation to use a *lower case* letter for vectors (along with **bold**, vector *etc.* of course)

Bold upper case letters are reserved for matrices — **more later**.

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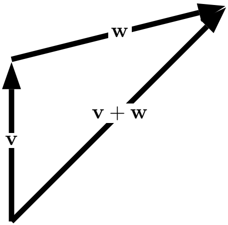
Definition (The triangle law)

We add two vectors $\mathbf{v}$ and $\mathbf{w}$ in the following way.

We arrange $\mathbf{w}$ such that its tail coincides with the head of $\mathbf{v}$.

$\mathbf{u} = \mathbf{v} + \mathbf{w}$ is then defined as the vector with the tail of $\mathbf{v}$ and the head of the newly arranged vector $\mathbf{w}$.

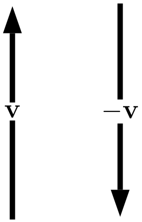


Definition (**0** and opposite vectors)

We define $\mathbf{0}$ as the vector with length 0.

If $\mathbf{v}$ is not $\mathbf{0}$, then we define $-\mathbf{v}$ as the vector with the same length and the opposite direction as $\mathbf{v}$.

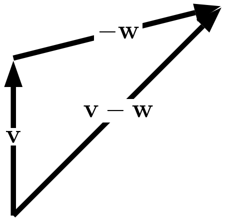
We see that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.



Definition (The difference of two vectors)

The difference of two vectors **v** and **w** is defined as

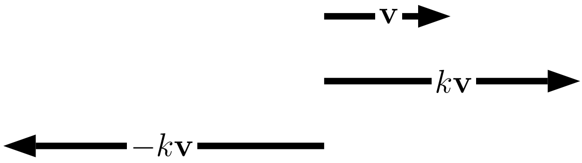
$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$



Definition (Scalar multiplication)

Let $\mathbf{v}$ be a vector and k a real number.

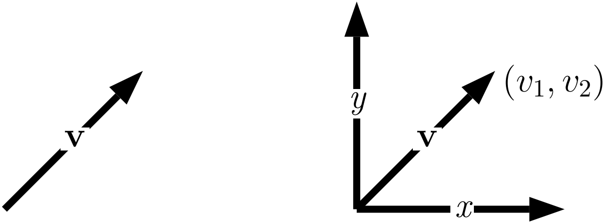
The vector $k\mathbf{v}$ is defined as the vector with the same direction as $\mathbf{v}$ if k is positive and the opposite direction if k is negative.



The length of $k\mathbf{v}$ is $|k|\times$ the length of $\mathbf{v}$.

Vectors in coordinate systems

We can simplify the analysis of vectors by introducing coordinate systems.
We consider the standard coordinate system in the $x - y$ (2D $\mathbb{R}^2$) plane:



If **v** is a two-dimensional vector we can always arrange it such that its tail coincides with the origin.

The coordinates (v_1, v_2) of its head uniquely identify **v** and are called the components of **v**.

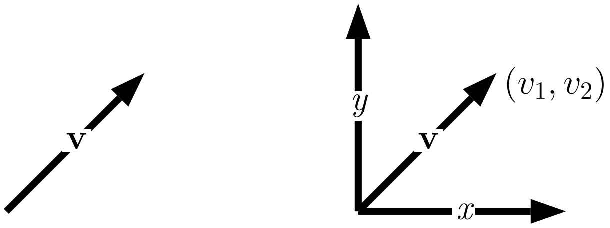
Vector Definition in $\mathbb{R}^2$

Since the coordinates of the head determine any vector uniquely, we make the following definition:

Definition (Vectors in the space $\mathbb{R}^2$)

We identify the space of two-dimensional vectors with

$$\mathbb{R}^2 = \{(x, y) |\ x \in \mathbb{R},\ \ y \in \mathbb{R}\}$$

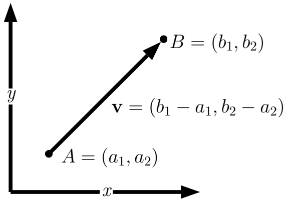


We now write the notation $\mathbf{v}$ for the vector (v_1, v_2).

Calculating Vectors from 2 Points in $\mathbb{R}^2$

If a vector $\mathbf{v}$ is defined by two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ we can get the components of $\mathbf{v}$ by the simple calculation:

$$\mathbf{v} = (b_1 - a_1, b_2 - a_2) \quad \text{Head - Tail}$$



Thus two vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ are **equivalent** if $v_1 = w_1$ **and** $v_2 = w_2$.

Calculating Vectors from 2 Points in $\mathbb{R}^2$ using MATLAB

If a vector **v** is defined by two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ we can get the components of **v** in MATLAB:

```
% Symbolic
>> syms a1 a2 b1 b2

>> A=[a1, a2]
A =
[ a1, a2]

>> B=[b1, b2]
B =
[ b1, b2]

>> AB=B-A
AB =
[ b1 - a1, b2 - a2]
```

```
% assume A(-1, 2), B(3, 5)

>> A=[-1, 2]
A =
    -1     2

>> B=[3, 5]
B =
     3     5

>> AB=B-A
AB =
     4     3
```

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Vectors in Higher Dimensional Spaces

Since we identified the space of two-dimensional vectors with the space of all ordered 2-tuples we can define higher dimensional vector spaces in the same way.

Definition (Vectors in the spaces $\mathbb{R}^3$ and $\mathbb{R}^n$)

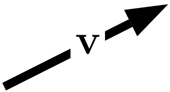
We define the space of three-dimensional vectors as

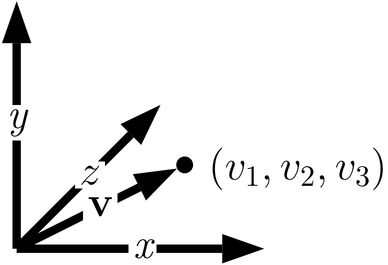
$$\mathbb{R}^3 = \{(x, y, z)|\ x \in \mathbb{R},\ \ y \in \mathbb{R},\ \ z \in \mathbb{R}\}$$

Let n be a positive integer. We define the space of n-dimensional vectors as

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n)|\ x_1 \in \mathbb{R},\ \ x_2 \in \mathbb{R},\ \ \ldots,\ \ x_n \in \mathbb{R}\}$$

Standard coordinate system in $\mathbb{R}^3$





Vector operators

Definition (Vector Addition, Subtraction and Scalar Multiplication in $\mathbb{R}^n$)

Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $\mathbb{R}^n$ and k a real number. The following rules are well-defined:

- $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$.
- $\mathbf{v} - \mathbf{w} = (v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n)$
- $k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$.

These rules coincide with the geometrical interpretation for two-dimensional vectors (see previous definitions).

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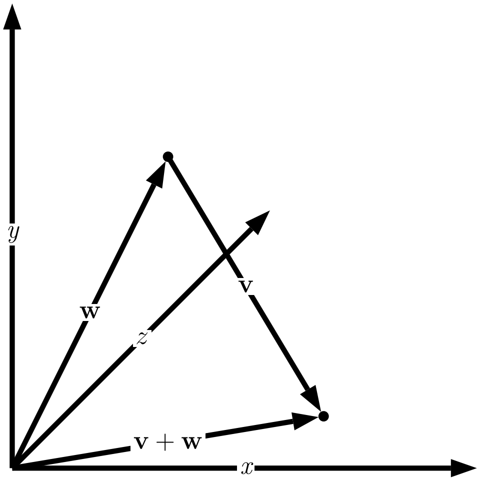
Example (Vector Addition, Subtraction and Scalar Multiplication in $\mathbb{R}^n$)



Vector Addition, Subtraction and Scalar Multiplication in MATLAB

MATLAB directly supports vector addition, subtraction and scalar multiplication:

```
>> v=[1 2 5];
>> w=[3 -1 1];
>> v+w
ans =
     4     1     6

>> v-w
ans =
    -2     3     4

>> 3 * v
ans =
     3     6    15

>> w*(-1)
ans =
    -3     1    -1
```

Scalar product

Definition (Scalar product)

Given two vectors v and w in $\mathbb{R}^n$ with components $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$. We define the *scalar product* (or *(standard) inner product*, *dot product*) of $\mathbf{v}$ and $\mathbf{w}$ as

$$\mathbf{v}.\mathbf{w} \;\text{ or }\; \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i w_i$$

Note what the scalar product does:

It takes two vectors and assigns them a real number.

Problem (Scalar product)

Work out the scalar product of vectors $\mathbf{v} = (1, 2)$ and $\mathbf{w} = (2, 3)$

Note the notations $\mathbf{v}.\mathbf{w}$ and $\langle \mathbf{v}, \mathbf{w} \rangle$ are equivalent.
We use the $\mathbf{v}.\mathbf{w}$ notation.

Scalar product using MATLAB

MATLAB provides a vector function `dot` that computes the dot product of two vectors (of any, identical dimension).

```
>> v = [3 2 -1]
>> w = [2 -1 1]
>> dot(v, w)
ans =
     3
>> sum(v.*w)
ans =
     3
```

`dot(v, w)` is equivalent to `sum(v.*w)` note `v.*w` is an array multiplication that returns a vector of the same size.

Theorem (Scalar product properties)

The scalar product has the following properties.

Theorem (Scalar product properties)

- $\mathbf{v}.\mathbf{v} \geq 0$, *for all* $\mathbf{v} \in \mathbb{R}^n$ *and* $\mathbf{v}.\mathbf{v} = 0 \iff v = 0$.
- $\mathbf{v}.\mathbf{w} = \mathbf{w}.\mathbf{v}$, *for all* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
- $(\mathbf{v} + \mathbf{u}).\mathbf{w} = \mathbf{v}.\mathbf{w} + \mathbf{u}.\mathbf{w}$, *for all* $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^n$.
- $(k\mathbf{v}).\mathbf{w} = k(\mathbf{v}.\mathbf{w})$ *for all* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $k \in \mathbb{R}$.

Euclidean norm of a vector

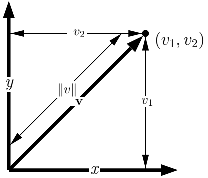
Definition (Euclidean norm of a vector)

For a vector $\mathbf{v} \in \mathbb{R}^n$ we define its norm as

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}.\mathbf{v}}$$

This norm is called the Euclidean norm of the vector $\mathbf{v}$.

The Euclidean norm of a vector coincides with the length of the vector in $\mathbb{R}^2$ and $\mathbb{R}^3$.



By Pythagoras' Theorem, $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{\mathbf{v}.\mathbf{v}}$

Euclidean norm of a vector in MATLAB

The default behaviour of MATLAB function `norm` for a given vector input is to return the Euclidean norm (also called 2-norm):

```
>> v = [3 4]
>> norm(v)
ans =
     5
>> sqrt(dot(v, v))
ans =
     5
```

Properties of scalar products

Theorem (Cauchy-Schwarz inequality)

Let v and w be vectors in $\mathbb{R}^n$
Then they satisfy the Cauchy-Schwarz inequality

$$|\mathbf{v}.\mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|.$$

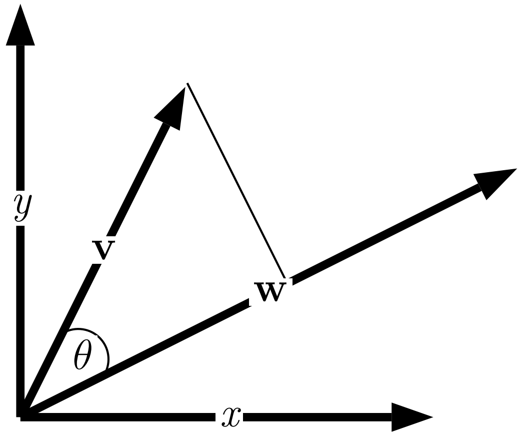
Theorem (Angle Between Two Vectors)

If $n = 2, 3$ we even have the relation

$$\mathbf{v}.\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos(\theta)$$

We call θ the angle between v and w.

Geometric Visualisation of Angle Between Two Vectors in $\mathbb{R}^2$



Properties of scalar products

Proof of Cauchy-Schwarz inequality in $\mathbb{R}^n$

Let $\mathbf{v} = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ and $\mathbf{w} = (\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n})$ be two vectors in $\mathbb{R}^n$, the quadratic function of z:

$$\sum_{i=1}^{n}(v_i z - w_i)^2 = (v_1 z - w_1)^2 + (v_2 z - w_2)^2 + \cdots + (v_n z - w_n)^2 = 0$$

can have at most one solution. Denote this as $az^2 + bz + c = 0$ where $a = v_1^2 + v_2^2 + \cdots + v_n^2$, $b = -2(v_1 w_1 + v_2 w_2 + \cdots + v_n w_n)$, c$= w_1^2 + w_2^2 + \cdots + w_n^2$.
So the discriminant

$$b^2 - 4ac = 4(\mathbf{v}.\mathbf{w})^2 - 4\|\mathbf{v}\|^2\|\mathbf{w}\|^2 \leq 0$$

Therefore $|\mathbf{v}.\mathbf{w}| \leq |\mathbf{v}\|\|\mathbf{w}\|$.

Properties of scalar products

Proof of Angle Between Two Vectors in $\mathbb{R}^n$

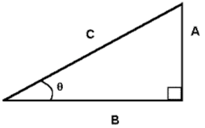
Let $\mathbf{v} = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ and $\mathbf{w} = (\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n})$ be two vectors in $\mathbb{R}^n$,

$$\mathbf{v}.\mathbf{w} = \sum_{i=1}^{n} v_i w_i = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

We need to find out what $\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$ is.

Basic Trigonometric Formulae / Pythagoras' Theorem

We review some simple trigonometry here.
For a right-angle triangle

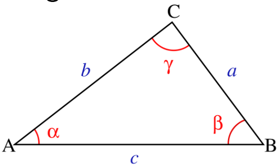


$\sin\theta = A/C$, $\cos\theta = B/C$ and $\tan\theta = A/B$
Also Pythagoras' Theorem states that

$$A^2 + B^2 = C^2$$

Law of Consines

A generalisation of Pythagoras's Theorem:



$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

If $\gamma = 90^\circ$, $\cos\gamma = 0$, this is equivalent to Pythagoras' Theorem.

Properties of scalar products

Proof of Angle Between Two Vectors in $\mathbb{R}^n$ (cont.)

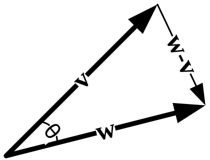
According to Law of Cosines,

$$\|\mathbf{w}-\mathbf{v}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta,$$

$$\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = \frac{1}{2}\left(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{w}-\mathbf{v}\|^2\right)$$

Note $\left(v_i^2 + w_i^2 - (w_i - v_i)^2\right)/2 = v_i w_i$, we have

$$\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = \sum_{i=1}^{n} v_i w_i = \mathbf{v}.\mathbf{w}.$$



Example (Orthogonal vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$)

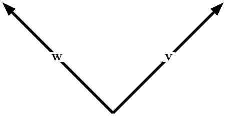
Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ be two vectors in $\mathbb{R}^2$.

We call $\mathbf{v}$ and $\mathbf{w}$ **orthogonal** if the angle between them is $90°$.

Since $\cos(\theta) = 0$ **if and only if** $\theta = 90°$ for $\theta \in [0, 180°]$ we can conclude that orthogonal vectors are characterized by the relation

$$\mathbf{v}.\mathbf{w} = 0.$$

This expression is also meaningful in $\mathbb{R}^n$ and we say that two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ are **orthogonal** , **if** their *scalar product* is **zero**.



The Vector Cross Product

Besides the scalar product that maps two vectors from $\mathbb{R}^n$ to $\mathbb{R}$ we also need a product that maps two vectors from $\mathbb{R}^n$ to a vector in $\mathbb{R}^n$.

Definition (The vector cross product in $\mathbb{R}^2$)

We define the **vector cross product** of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ as a mapping $\times : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ with

$$\mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1$$

The vector product in $\mathbb{R}^2$ is anti-symmetric, i.e.

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$$

The Vector Cross Product (cont.)

Definition (The vector cross product in $\mathbb{R}^3$)

We define the vector cross product of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ as a mapping $\times : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ with

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}.$$

The vector product in $\mathbb{R}^3$ is also anti-symmetric, i.e.

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$$

The vector cross product has very useful properties, especially:

- for finding orthogonal vectors in $\mathbb{R}^3$.
- for area and volume calculations in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Example (Vector cross product: Orthogonal Vectors)

Work out the vector cross product of the vectors $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

It is easy to show that:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

Example (Vector cross product: parallel vectors)

Let $n = 2$.

If $\mathbf{v} = (v_1, v_2)$ then the vector $\mathbf{v}^{perp} = (-v_2, v_1)$ is orthogonal to v.

It follows from the definition of the vector and scalar product, that

$$\mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1 = -v_2 w_1 + v_1 w_2 = \mathbf{v}^{perp}.\mathbf{w}$$

This expression is 0 if $\mathbf{w}$ and $\mathbf{v}^{perp}$ are **orthogonal**.

However this means that $\mathbf{v}$ and $\mathbf{w}$ are **parallel**.

Vector cross products in MATLAB

MATLAB provides a vector function `cross` to compute the cross product of two vectors in $\mathbb{R}^3$:

```
>> v=[1 2 3];
>> w=[-1 1 2];
>> cross(v, w)
ans =
     1     -5     3
```

Theorem (Parallel vectors in $\mathbb{R}^2$)

We call two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$ *parallel* if we have $\mathbf{v} \times \mathbf{w} = 0$.

We even have

$$\mathbf{v} \times \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$ counted positive counter-clockwise and negative clockwise starting from $\mathbf{v}$.

Can you prove this?

Generalisation of sinusoidal relation

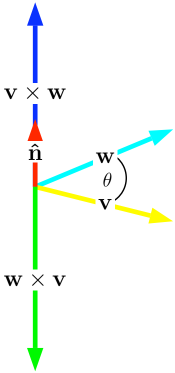
In general for any dimension it can be stated that:

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)$$

We also have:

$$\mathbf{v} \times \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector (of length 1) perpendicular to both **v** and **w**



Generalisation of sinusoidal relation (cont.)

We prove this using MATLAB symbolic toolbox:

To prove the first equation, note that $\mathbf{v}.\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$, it is sufficient to show if

$$\|\mathbf{v} \times \mathbf{w}\|^2 + |\mathbf{v}.\mathbf{w}|^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2.$$

(since $\sin^2\theta + \cos^2\theta = 1$)

```
>> syms v1 v2 v3 real
>> syms w1 w2 w3 real
>> v=[v1 v2 v3];
>> w = [w1 w2 w3];
>> f=dot(cross(v, w),cross(v, w))+dot(v, w).^2
f =
(v1*w2 - v2*w1)^2 + (v1*w3 - v3*w1)^2 + (v2*w3 - v3*w2)^2
  + (v1*w1 + v2*w2 + v3*w3)^2
>> simplify(f)
ans =
(v1^2 + v2^2 + v3^2)*(w1^2 + w2^2 + w3^2)
```

Generalisation of sinusoidal relation (cont.)

We prove this using MATLAB symbolic toolbox:

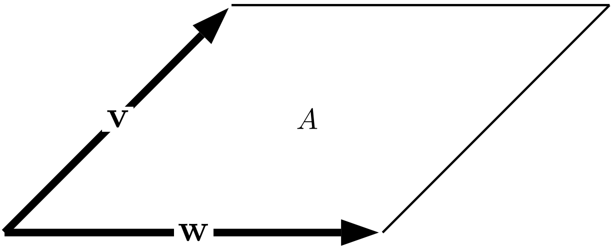
To prove the second equation, we need to verify that **v**.**w** is orthogonal to both **v** and **w**:

```
>> syms v1 v2 v3 real
>> syms w1 w2 w3 real
>> v=[v1 v2 v3];
>> w = [w1 w2 w3];
>> f1=dot(cross(v, w),v)
f1 =
v3*(v1*w2 - v2*w1) - v2*(v1*w3 - v3*w1) + v1*(v2*w3 - v3*w2)
>> simplify(f1)
ans =
0
>> f2=dot(cross(v, w),w)
f2 =
w3*(v1*w2 - v2*w1) - w2*(v1*w3 - v3*w1) + w1*(v2*w3 - v3*w2)
>> simplify(f2)
ans =
0
```

Back to our Volume Calculation

If a parallelogram is spanned by $\mathbf{v}$ and $\mathbf{w}$ then its area A is given by

$$A = |\mathbf{v} \times \mathbf{w}|.$$



Volume Calculation in $\mathbb{R}^3$

Now let $n = 3$ and let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$.

We have similar relationships as in the case $n = 2$.

One can show that

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\||\sin\theta|$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

In particular $\mathbf{v}$ and $\mathbf{w}$ are **parallel** **only if**

$$\mathbf{v} \times \mathbf{w} = \mathbf{0} \quad \mathbf{0} \text{ is the zero vector in } \mathbb{R}^3.$$

Volume Calculation in $\mathbb{R}^3$: Scalar product/Cross Product

Now consider the expression: $\mathbf{v}.(\mathbf{v} \times \mathbf{w})$

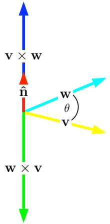
It holds that

$$\mathbf{v}.(\mathbf{v}\times\mathbf{w}) = v_1(v_2w_3 - v_3w_2) + v_2(v_3w_1 - v_1w_3) + v_3(v_1w_2 - v_2w_1) = 0$$

Similarly we can show

$$\mathbf{w}.(\mathbf{v} \times \mathbf{w}) = 0$$

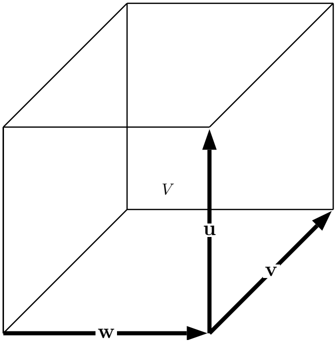
and we have seen that the vector $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.



Volume in $\mathbb{R}^3$: A parallelepiped

As in the two-dimensional case we get an easy formula for the volume of a parallelepiped spanned by three vectors **v**, **w** and **u**.

$$V = |\mathbf{v}.(\mathbf{w} \times \mathbf{u})| = |\mathbf{w}.(\mathbf{v} \times \mathbf{u})| = |\mathbf{u}.(\mathbf{v} \times \mathbf{w})|$$



Example (Volume Worked Example in $\mathbb{R}^2$)

The area of the parallelogram spanned by the two vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is given by

$$A = \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right| = \|1 \cdot 0 - 1 \cdot 2\| = 2$$

Example (Volume Worked Example in $\mathbb{R}^3$)

The volume, V, of the parallelepiped spanned by the three vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$ is given by:

$$\begin{aligned} V &= \left| \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} . \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right) \right| \\ &= \left| \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right| \\ &= |-4| \\ &= 4 \end{aligned}$$

Identities for the vector and the scalar product

Theorem (Identities for the vector and the scalar product)

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *and* $\mathbf{x} \in \mathbb{R}^3$. *Then we have the following identities.*

- $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.
- $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\||\sin(\theta)|$.
- $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{u}.\mathbf{v})\mathbf{w}$ *(Grassmann-expansion).*
- $(\mathbf{u} \times \mathbf{v}).(\mathbf{w} \times \mathbf{x}) = (\mathbf{u}.\mathbf{w})(\mathbf{v}.\mathbf{x}) - (\mathbf{v}.\mathbf{w})(\mathbf{u}.\mathbf{x})$. *(Lagrange identity).*

You can prove them by using the definitions of cross and scalar products and expand the equations. Using MATLAB can save a lot of tedious calculation.

Summary

We have discussed

- 2, 3 and n-dimensional vectors.
- Representing vectors in coordinate systems.
- Vector addition, subtraction and scalar multiplication.
- Vector scalar and cross products and their properties.
- Using MATLAB to calculate vector operations and verify/prove properties.