

CM2202: Scientific Computing and Multimedia Applications

Geometric Computing: 1. Introduction

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Geometric Computing

We will look at **Geometric Computing** in this section.

The fundamental basics of:

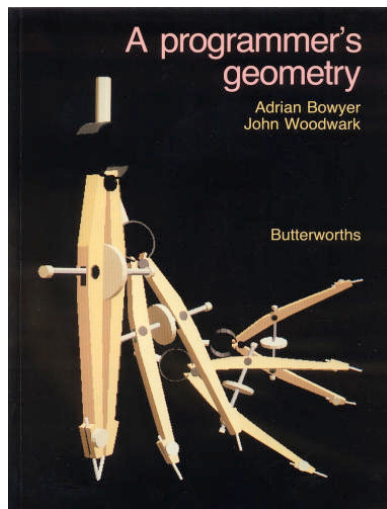
- Computer Graphics
- Image Processing and Computer Vision
- Spatial Reasoning, Geographic Information Systems.

Builds on Linear Algebra:

- Vectors, Matrices
- Linear Equations

References

A Programmer's Geometry,
Adrian Bowyer,
John Woodwark,
Butterworths, 1983,
ISBN: 0408012420.



[MATLAB Geometry Toolbox](#)

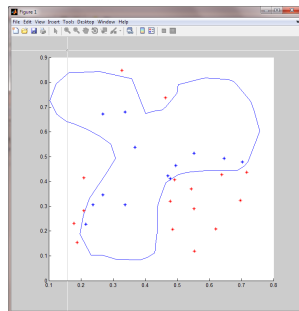
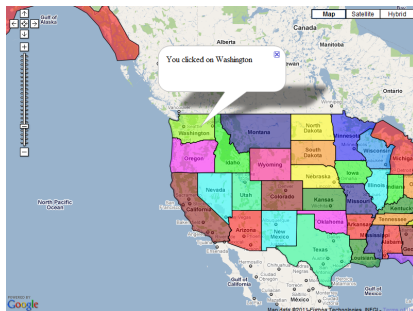
Example Applications

We show some practical application scenarios of geometric computing and some demos. These are only some **examples** and there are many more possibilities:

- Geographic Information Systems: Point Location
- Geometric Modelling: Spline Fitting
- Computer Graphics: Ray Tracing
- Image Processing: Hough Transform
- Mobile Systems: Spatial Location Sensing

Example Application: Point Location in GIS etc.

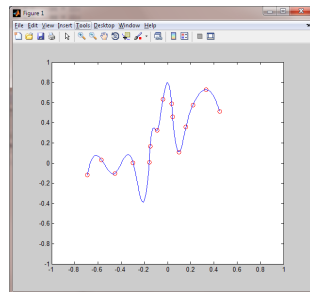
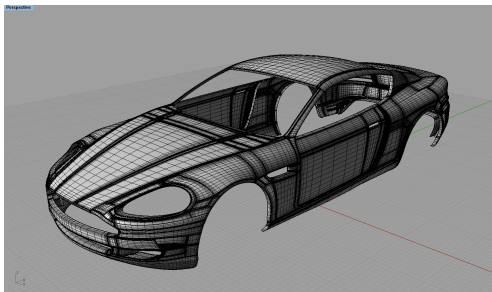
GIS involve lots of geometric primitives and their interactions. A simple example is testing if a point locates within a certain region (often modelled as a polygon). This can be used to find where you are from GPS data or identify the region that the user clicks.



Interactive Map Demo

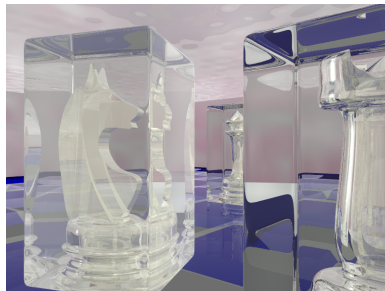
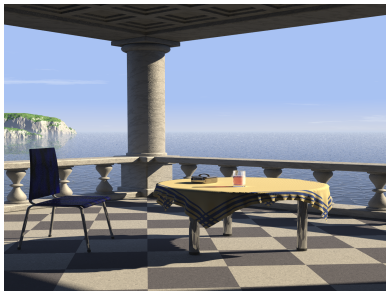
Example Application: Spline Fitting in Geometric Modelling

Geometric modelling provide tools that help design and manufacture of products (e.g. cars, airplanes, garments etc.)
Spline (piecewise polynomial curves and surfaces) is a fundamental technique.



Example Application: Ray Tracing in Computer Graphics

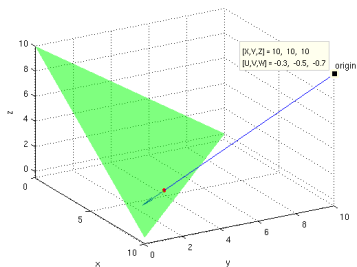
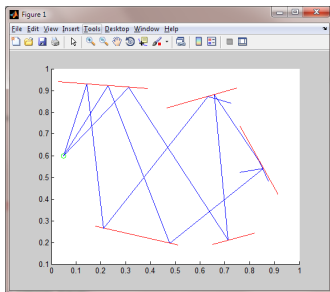
Computer graphics aim at reproducing or creating vivid animations in computers. Ray tracing is a widely used technique for generating high-quality rendering of virtual scenes.



Rendered with POV-RAY

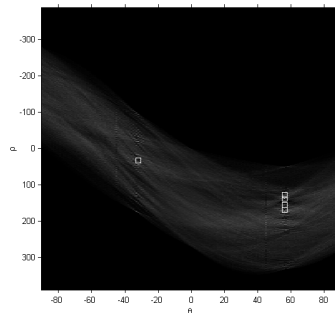
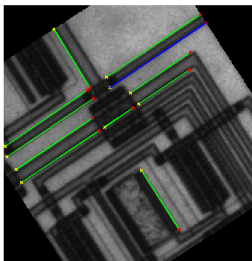
Ray Tracing in Computer Graphics (cont.)

Ray object intersection is the key operation in ray tracing algorithm. Some demos:



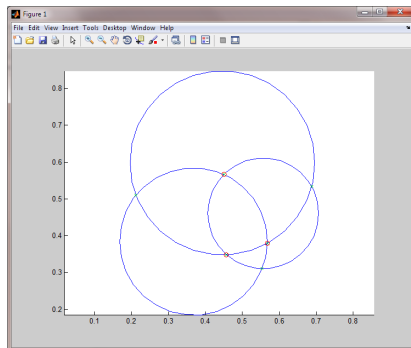
Example Application: Hough Transform in Image Processing / Computer Vision

Computer Vision considers the inverse problem of “understanding” images. To identify some significant structures from images is needed by many application scenarios. Hough transform is used to find prominent features (lines, circles etc.) from images, using some voting scheme in the implicit parameter space.



Example Application: Spatial Location Sensing in Mobile Systems

With techniques such as RFID, 3D location sensing is possible. Multiple sources of information can be combined, potentially with some uncertainty. A simple 2D demo involves circle to circle intersection to identify the common region suggested by multiple sensors.

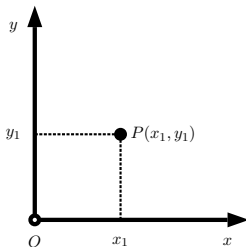


Coordinate Systems: 2D

The **Cartesian coordinate system** (also called *rectangular coordinate system*) determines each point uniquely in a plane through two numbers, usually called

- the **x-coordinate** or **abscissa**
- the **y-coordinate** or **ordinate** of the point.

with respect to two orthogonal axes, the x-axis and y axis.

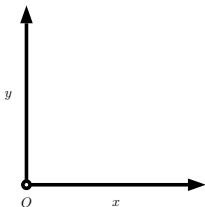


2D Coordinate Systems: Handedness

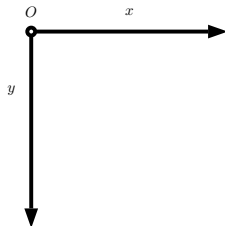
We can draw our coordinate system in one of two ways.

Fixing the x -axis to point horizontally from left to right, we can draw the y -axis in one of two ways:

y -axis pointing up vertically



y -axis pointing down vertically



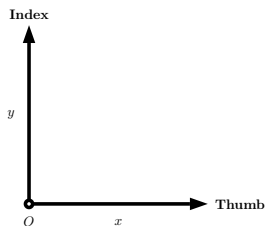
Also called the **positive** or **standard** orientation

2D Coordinate Systems: Right/Left Handedness

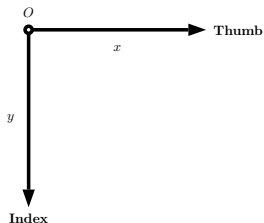
An easy way to define and remember each coordinate system is to **use your hands**:

- Assign your **thumb** to the **x-axis**
- Assign your **index finger** to the **y-axis**
- Right or left hand will align with axes accordingly (Palm facing towards you).

Right Handed System

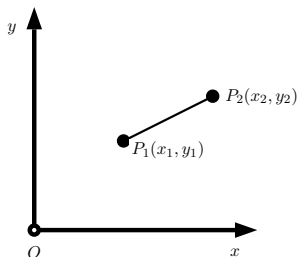


Left Handed System

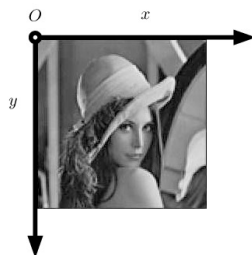


2D Coordinate Systems: Handedness Examples

Right Handed System:
Standard Graph Plotting



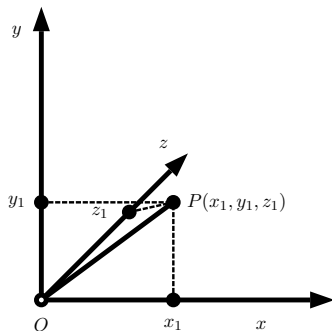
Left Handed System:
Image Pixel Coordinate Indexing



3D Coordinate Systems

3D coordinates systems build on similar ideas to the previous 2D systems, we now need to account for the **third dimension** — the ***z*-axis**.

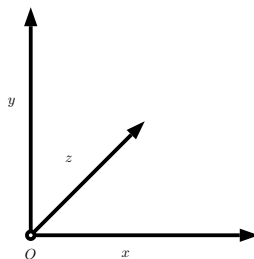
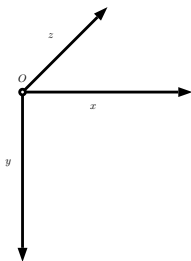
All three axes are orthogonal (perpendicular) to each other.



3D Coordinate Systems: Handedness

As with 2D, we can draw our coordinate system in one of two ways.

Once the x - and y -axes are specified, they determine the line along which the z -axis should lie, but there are two possible directions on this line:

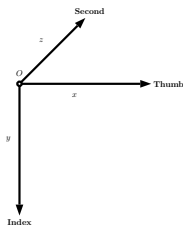


3D Coordinate Systems: Right/Left Handedness

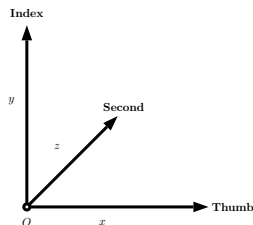
Again **use your hands**:

- Assign your **thumb** to the **x-axis**
- Assign your **index finger** to the **y-axis**
- Assign your **second finger** to the **z-axis**
- Right or left hand will align with axes accordingly (sometimes with some contortion!).

Right Handed System



Left Handed System



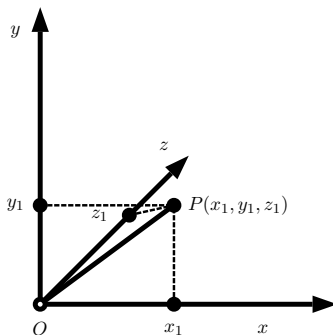
3D Coordinate Systems: Handedness Examples

Right Handed System:

Video Stack Indexing:
z-axis is time



Left Handed System:

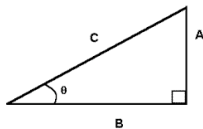


Mathematical Tools Recap

We review some simple mathematical tools used throughout the session.

Basic Trigonometric Formulae / Pythagoras' Theorem

For a right-angle triangle



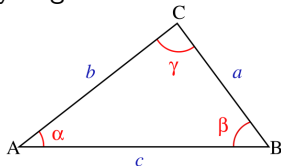
$$\sin \theta = A/C, \cos \theta = B/C \text{ and } \tan \theta = A/B$$

Also Pythagoras' Theorem states that

$$A^2 + B^2 = C^2$$

Law of Cosines

A generalisation of Pythagoras's Theorem:



$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

If $\gamma = 90^\circ$, $\cos \gamma = 0$, this is equivalent to Pythagoras' Theorem.

Basic Linear Algebra/Vector Formulae

For two 3D vectors $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$.

$$\mathbf{v}_1 \pm \mathbf{v}_2 = (x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2)$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2$$

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1).$$

Matrix operations (addition, subtraction, multiplication and division).

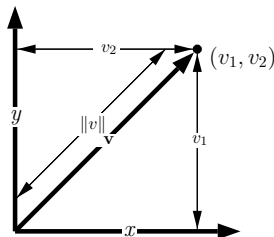
Euclidean norm of a vector

For a vector $\mathbf{v} \in \mathbb{R}^n$ we define its norm as

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

This norm is called the Euclidean norm of the vector \mathbf{v} .

The Euclidean norm of a vector coincides with the length of the vector in \mathbb{R}^2 and \mathbb{R}^3 .



By Pythagoras' Theorem, $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

Cauchy-Schwarz inequality

Let v and w be vectors in \mathbb{R}^n

Then they satisfy the Cauchy-Schwarz inequality

$$\mathbf{v} \cdot \mathbf{w} \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

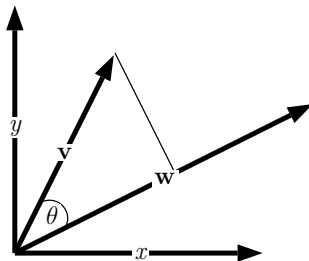
Angle Between Two Vectors

If $n = 2, 3$ we even have the relation

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

We call θ the *angle between v and w* .

Geometric Visualisation of Angle Between Two Vectors in \mathbb{R}^2



Variable Substitution

To ease algebraic manipulation in deriving equations it may be useful to group variables together by substituting the group for a single variable. This may be replaced later in the derivation if needed.

For example it is far easier to expand $(x + x_t)^2$ rather than $(x + x_a + x_b + x_c)^2$.

Here we simply let $x_t = x_a + x_b + x_c$

Quadratic Equations

If $ax^2 + bx + c = 0$ then the roots of x are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Determinants

A determinant is a number. A determinant is evaluated by scanning along one of its rows or columns and alternately adding and subtracting the value of the determinant formed by omitting the row and column corresponding to the value multiplied by that value.

A second order determinant

$$\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} = d_{11}d_{22} - d_{12}d_{21}$$

A third order determinant

$$\begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} \\ = d_{11}(d_{22}d_{33} - d_{32}d_{23}) - d_{12}(d_{21}d_{33} - d_{31}d_{23}) + d_{13}(d_{21}d_{32} - d_{31}d_{22}).$$

Linear Equations

For a linear system with n unknowns, x_1, x_2, \dots, x_n , to have a unique solution, n independent linear equations are needed:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

...

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Write in matrix form with $\mathbf{A} = (a_{ij})_{n \times n}$, $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)^T$:

$$\mathbf{Ax} = \mathbf{b}.$$

The system has unique answer if and only if $\det \mathbf{A} \neq 0$. The solution is

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{\text{adj}\mathbf{A}}{\det \mathbf{A}}\mathbf{b}$$

Apply Other Geometric Formulae

Many of the simpler derivations, such as perpendicular distance of a point to a line or derivation of a line equation, are used in more involved derivations.

Know your core Geometric derivations