CM2202: Scientific Computing and Multimedia Applications
Geometric Computing: 3. Lines and Surfaces in 3D

Dr. Yukun Lai

School of Computer Science & Informatics

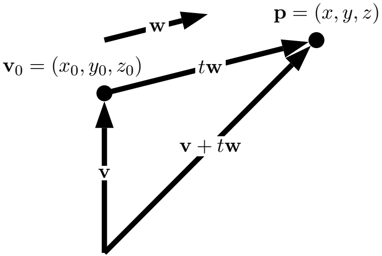
Lines in 3D

In 3D the **implicit equation** of a line is defined as the **intersection of two planes**. (More on this shortly)

The **parametric equation** is a simple extension to 3D of the 2D form:

$$\begin{aligned} x &= x_0 + ft \\ y &= y_0 + gt \\ z &= z_0 + ht \end{aligned}$$

Parametric Lines in 3D



$$
\begin{aligned}
x &= x_0 + ft \\
y &= y_0 + gt \\
z &= z_0 + ht
\end{aligned}
$$

This is simply an **extension** of the vector form in 3D

The line is normalised when $f^2 + g^2 + h^2 = 1$

Perpendicular Distance from a Point to a Line in 3D

For the parametric form,

$$\begin{aligned} x &= x_0 + ft \\ y &= y_0 + gt \\ z &= z_0 + ht \end{aligned}$$

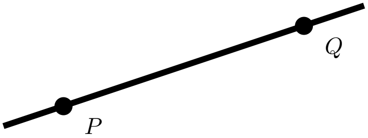
This builds on the 2D example we met earlier, line_par_point_dist_2d. The 3D form is line_par_point_dist_3d.

```
dx =   g * ( f * ( p(2) - y0 ) - g * ( p(1) - x0 ) ) ...
     + h * ( f * ( p(3) - z0 ) - h * ( p(1) - x0 ) );

dy =   h * ( g * ( p(3) - z0 ) - h * ( p(2) - y0 ) ) ...
     - f * ( f * ( p(2) - y0 ) - g * ( p(1) - x0 ) );

dz = - f * ( f * ( p(3) - z0 ) - h * ( p(1) - x0 ) ) ...
     - g * ( g * ( p(3) - z0 ) - h * ( p(2) - y0 ) );

dist = sqrt ( dx * dx + dy * dy + dz * dz ) ...
    / ( f * f + g * g + h * h );
```

The value of parameter, t, where the point intersects the line is given by:

```
t =  (f*( p(1) - x0) + g*(p(2) - y0) + h*(p(3) - z0)/( f * f + g * g + h*h);
```

Line Through Two Points in 3D (parametric form)



The parametric form of a line through two points, $P(x_p, y_p, z_p)$ and $Q(x_q, y_q, z_q)$ comes readily from the vector form of line (again a simple extension from 2D):

- Set base to point P
- Vector along line is $(x_q - x_p, y_q - y_p, z_q - z_p)$
- The equation of the line is:

$$
\begin{aligned}
x &= x_p + (x_q - x_p)t \\
y &= y_p + (y_q - y_p)t \\
z &= z_p + (z_q - z_p)t
\end{aligned}
$$

- As in 2D, $t = 0$ gives P and $t = 1$ gives Q
- Normalise if necessary.

Plot 3D lines in MATLAB

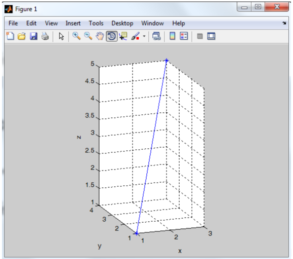
To plot a line segment with end points (x_1, y_1, z_1) and (x_2, y_2, z_2), you can use `plot3([x1 x2], [y1 y2], [z1 z2]);` (similar to `plot` in 2D – see `help plot3`).

Example: To plot a line segment from $(1, 1, 1)$ to $(3, 4, 5)$:

```
>> plot3([1 3], [1 4], [1 5], '*-');
```

To make the 3D line more clearly visible, you may enable the grid and add labels to x-/y-/z-axes.

```
>> grid on; axis equal; xlabel('x'); ylabel('y'); zlabel('z');
```



Implicit Surfaces

An implicit surface (just like implicit curves in 2D) of the form

$$f(x, y, z) = 0$$

We simply add the **extra z dimension**.

For example:

- A plane can be represented

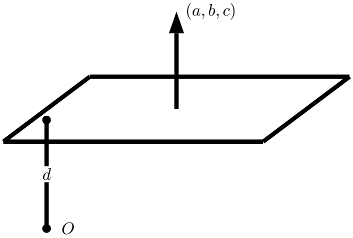
$$ax + by + cz + d = 0$$

- A sphere can be represented as

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0$$

  which is just the extension of the circle in 2D to 3D where the centre is now (x_c, y_c, z_c) and the radius is r.

Implicit Equation of a Plane

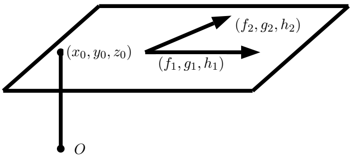


The plane equation:

$$ax + by + cz + d = 0$$

- Is normalised if $a^2 + b^2 + c^2 = 1$.
- Like 2D the **normal vector** — **the surface normal** — is given by a vector $\mathbf{n} = (a, b, c)$
  - a, b and c are the cosine angles which the normal makes with the x-,y- and z-axes respectively.

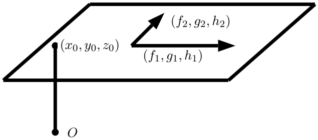
Parametric Equation of a Plane



$$
\begin{aligned}
x &= x_0 + f_1 u + f_2 v \\
y &= y_0 + g_1 u + g_2 v \\
z &= z_0 + h_1 u + h_2 v
\end{aligned}
$$

- This is an extension of parametric line into 3D where we now have two variable parameters u and v that vary.
- (f_1, g_1, h_1) and (f_2, g_2, h_2) are two **different** vectors **parallel** to the plane.

Parametric Equation of a Plane (Cont.)

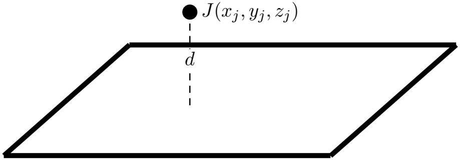


$$
\begin{aligned}
x &= x_0 + f_1 u + f_2 v \\
y &= y_0 + g_1 u + g_2 v \\
z &= z_0 + h_1 u + h_2 v
\end{aligned}
$$

- A point in the plane is found by adding proportion u of one vector to a proportion v of the other vector
- If the two vectors are of unit length and are perpendicular, then:

$$
\begin{aligned}
f_1^2 + g_1^2 + h_1^2 &= 1 \\
f_2^2 + g_2^2 + h_2^2 &= 1 \\
f_1 f_2 + g_1 g_2 + h_1 h_2 &= 0 \ \text{(scalar product)}
\end{aligned}
$$

Distance from a 3D point and a Plane



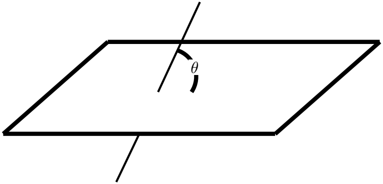
The distance, d, between a point, $J(x_j, y_j, z_j)$, and an **implicit** plane, $ax + by + cz + d = 0$ is:

$$d = \frac{ax_j + by_j + cz_j + d}{\sqrt{a^2 + b^2 + c^2}}$$

This is very similar to the 2D distance of a point to a line.
The MATALB code to achieve this is, plane_imp_point_dist_3d.m:

```
norm = sqrt ( a * a + b * b + c * c );
if ( norm == 0.0 )
      error ( 'PLANE Normal = 0!' );
end
dist = abs ( a * p(1) + b * p(2) + c * p(3) + d ) / norm;
```

Angle Between a Line and a Plane



If the plane is in implicit form $ax + by + cz + d = 0$ and line is in parametric form:

$$\begin{aligned} x &= x_0 + ft \\ y &= y_0 + gt \\ z &= z_0 + ht \end{aligned}$$

then the angle, γ between the line and the normal the plane (a, b, c) is:

$$\gamma = \cos^{-1}(af + bg + ch)$$

The angle, θ, between the line and the plane is then:

$$\theta = \frac{\pi}{2} - \gamma$$

Angle Between a Line and a Plane (cont)

If either line or plane equations are not normalised the we must normalise:

$$\gamma = \cos^{-1}\frac{(af + bg + ch)}{\sqrt{(a^2 + b^2 + c^2)}}, \gamma = \cos^{-1}\frac{(af + bg + ch)}{\sqrt{(f^2 + g^2 + h^2)}}, \gamma = \cos^{-1}\frac{(af + bg + ch)}{\sqrt{(a^2 + b^2 + c^2)(f^2 + g^2 + h^2)}}$$

The angle, θ, is as before:

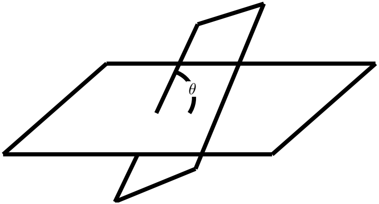
$$\theta = \frac{\pi}{2} - \gamma$$

The MATLAB code to so this is planes_imp_angle_line_3d.m:

```
norm1 = sqrt ( a1 * a1 + b1 * b1 + c1 * c1 );
 if ( norm1 == 0.0 )
    angle = Inf;
    return
 end

norm2 = sqrt ( f * f + g *g + h * h );
if ( norm2 == 0.0 )
    angle = Inf;
    return
  end

cosine = ( a1 * f + b1 * g + c1 * h) / ( norm1 * norm2 );
angle = pi/2 - acos( cosine );
```

CARDIFF UNIVERSITY
PRIFYSGOL CAERDYDD

Angle Between Two Planes



Given two **normalised** **implicit** planes

$$a_1x + b_1y + c_1z + d_1 = 0$$

and

$$a_2x + b_2y + c_2z + d_2 = 0$$

The angle between them, θ, is the angle between the normals:

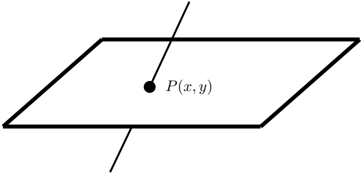
$$\theta = \cos^{-1}(a_1a_2 + b_1b_2 + c_1c_2)$$

Angle Between Two Planes (MATLAB Code)

The MATLAB code to do this is planes_imp_angle_3d.m:

```
norm1 = sqrt ( a1 * a1 + b1 * b1 + c1 * c1 );
if ( norm1 == 0.0 )
    angle = Inf;
    return
  end

norm2 = sqrt ( a2 * a2 + b2 * b2 + c2 * c2 );
if ( norm2 == 0.0 )
    angle = Inf;
    return
  end
cosine = ( a1 * a2 + b1 * b2 + c1 * c2 ) / ( norm1 * norm2 );
angle = acos ( cosine );
```

Intersection of a Line and Plane



If the plane is in implicit form $ax + by + cz + d = 0$ and line is in parametric form:

$$\begin{aligned} x &= x_0 + ft \\ y &= y_0 + gt \\ z &= z_0 + ht \end{aligned}$$

The the point, $P(x, y)$, where the is given by parameter, t:

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{af + bg + ch}$$

Intersection of a Line and Plane (MATLAB Code)

The MATLAB code to do this is plane_imp_line_par_int_3d.m

```
tol = eps;

norm1 = sqrt ( a * a + b * b + c * c );
if ( norm1 == 0.0 )
  error ( 'Norm1 = 0 - Fatal error!' )
end

norm2 = sqrt ( f * f + g * g + h * h );
if ( norm2 == 0.0 )
  error ( 'Norm2 = 0 - Fatal error!' )
end

denom = a * f + b * g + c * h;

if ( abs ( denom ) < tol * norm1 * norm2 ) %  The line and the plane may be parallel.
  if ( a * x0 + b * y0 + c * z0 + d == 0.0 )
    intersect = 1;
    p(1) = x0;
    p(2) = y0;
    p(3) = z0;
  else
    intersect = 0;
    p(1:dim_num) = 0.0;
  end
else
  intersect = 1;
  t = - ( a * x0 + b * y0 + c * z0 + d ) / denom; %  they must intersect.
  p(1) = x0 + t * f;
  p(2) = y0 + t * g;
  p(3) = z0 + t * h;
end
```

Intersection of Three Planes

- Three planes intersect at point.
- Two planes intersect in a line $\rightarrow$ two lines intersect at a point
- Similar problem to solving in 2D for two line intersecting:
  - Solve **three** simultaneous linear equations:

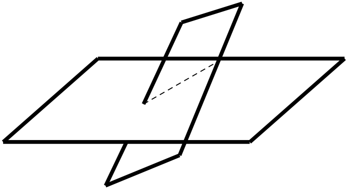
$$\begin{aligned} a_1 x + b_1 y + c_1 z + d_1 &= 0 \\ a_2 x + b_2 y + c_2 z + d_2 &= 0 \\ a_3 x + b_3 y + c_3 z + d_3 &= 0 \end{aligned}$$

Intersection of Three Planes (MATLAB Code)

The MATLAB code to do this is, planes_3d_3_intersect.m:

```
tol = eps;

bc = b2*c3 - b3*c2;
ac = a2*c3 - a3*c2;
ab = a2*b3 - a3*b2;


det = a1*bc - b1*ac + c1*ab

if (abs(det) < tol)
  error('planes_3d_3_intersct: At least to planes are parallel');
end;

else
 dc = d2*c3 - d3*c2;
 db = d2*b3 - d3*b2;
 ad = a2*d3 - a3*d2;

 detinv = 1/det;

 p(1) = (b1*dc - d1*bc - c1*db)*detinv;
 p(2) = (d1*ac - a1*dc - c1*ad)*detinv;
 p(3) = (b1*ad + a1*db - d1*ab)*detinv;

 return;
end;
```

Intersection of Two Planes



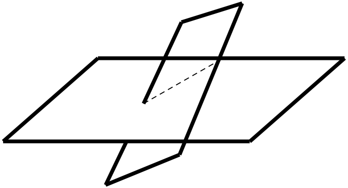
Two planes

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \end{aligned}$$

intersect to form a straight line:

$$\begin{aligned} x &= x_0 + ft \\ y &= y_0 + gt \\ z &= z_0 + ht \end{aligned}$$

Intersection of Two Planes (Cont.)



- (f, g, h) may be found by finding a vector along the line. This is given by the vector cross product of (a_1, b_1, c_1) and (a_2, b_2, c_2):

$$(f, g, h) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1 c_2 - b_2 c_1, a_2 c_1 - a_1 c_2, a_1 b_2 - a_2 b_1)$$

- (x_0, y_0, z_0) then readily follows.

Intersection of Two Planes (cont.)

$P_0(x_0, y_0, z_0)$ can be any point on the intersection line, so it should satisfy both plane equations:

$$a_1 x_0 + b_1 y_0 + c_1 z_0 + d_1 = 0$$

$$a_2 x_0 + b_2 y_0 + c_2 z_0 + d_2 = 0$$

Three unknowns (x_0, y_0 and z_0) but only two equations. Different possibilities for another equation. We choose the point on the intersection line that is closest to the origin. Geometrically $\mathbf{OP_0}$ is orthogonal to (f, g, h), so

$$fx_0 + gy_0 + hz_0 = 0$$

Intersection of Two Planes (cont.)

Write this in a matrix form

$$\begin{pmatrix} f & g & h \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ -d_1 \\ -d_2 \end{pmatrix}.$$

The determinant

$$\begin{vmatrix} f & g & h \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = f^2 + g^2 + h^2.$$

The solution can be found using MATLAB Symbolic Toolbox.

Intersection of Two Planes (Matlab Symbolic Toolbox)

```
>> syms a1 b1 c1 d1 a2 b2 c2 d2
>> v1 = [a1 b1 c1];
>> v2 = [a2 b2 c2];
>> v=cross(v1, v2)
v =
[ b1*c2 - b2*c1, a2*c1 - a1*c2, a1*b2 - a2*b1]
>> f = v(1);
>> g = v(2);
>> h = v(3);
>> A=[f g h; a1 b1 c1; a2 b2 c2]
A =
[ b1*c2 - b2*c1, a2*c1 - a1*c2, a1*b2 - a2*b1]
[            a1,            b1,            c1]
[            a2,            b2,            c2]
```

The determinant of **A**:

```
>> det(A)
ans =
a1^2*b2^2 + a1^2*c2^2 - 2*a1*a2*b1*b2 - 2*a1*a2*c1*c2 + a2^2*b1^2 + a2^2*c1^2 + b1^2*c2^2
 - 2*b1*b2*c1*c2 + b2^2*c1^2
>> simplify(det(A) - (f*f+g*g+h*h))
ans =
0
```

The solution to (x_0, y_0, z_0) is $\frac{1}{\det \mathbf{A}}$ times

```
>> simplify(A\b*det(A))
ans =
```

```
a1*b1*b2*d2-a2*b1∧2*d2-a1*c2∧2*d1-a2*c1∧2*d2-a1*b2∧2*d1+a2*b1*b2*d1+a1*c1*c2*d2+a2*c1*c2*d1
a1*a2*b1*d2-a1∧2*b2*d2-b1*c2∧2*d1-b2*c1∧2*d2-a2∧2*b1*d1+a1*a2*b2*d1+b1*c1*c2*d2+b2*c1*c2*d1
a1*a2*c1*d2-a1∧2*c2*d2-b2∧2*c1*d1-b1∧2*c2*d2-a2∧2*c1*d1+a1*a2*c2*d1+b1*b2*c1*d2+b1*b2*c2*d1
```

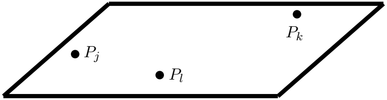
Intersection of Two Planes (Matlab Code)

The MATLAB code to do this is, planes_3d_2intersect_line.m:

```
tol = eps;

f = b1*c2 - b2*c1;
g = c1*a2 - c2*a1;
h = a1*b2 - a2*b1;


det = f*f + g*g + h*h;

if (abs(det) < tol)
  error('planes_3d_2intersect_line: Planes are parallel');
end;

else
 dc = d1*c2 - c1*d2;
 db = d1*b2 - b1*d2;
 ad = a1*d1 - a2*d1;

 detinv = 1/det;

 x0 = (g*dc - h*db)*detinv;
 y0 = -(f*dc +h*ad)*detinv;
 z0 = (f*db +g*ab)*detinv;

 return;
end;
```

Plane Through Three Points

Just as two points define a line, **three points** define a plane (this is the **explicit** form of a plane):



A plane may be found as follows:

- Let P_j be the point (x_j, y_j, z_j) in the plane
- Form two vectors in the plane $P_l - P_j$ and $P_k - P_j$
- Form another vector from a general point $P(x, y)$ in the plane to P_j
- The the equation of the plane is given by:

$$\begin{vmatrix} x - x_j & y - y_j & z - z_j \\ x_k - x_j & y_k - y_j & z_k - z_j \\ x_l - x_j & y_l - y_j & z_l - z_j \end{vmatrix} = 0$$

  (because the parallelepiped formed by these three vectors $P_l - P_j$, $P_k - P_j$ and $P - P_j$ should have zero volume.)
- a, b, c and d can be found by expanding the determinant above

Plane Through Three Points (MATLAB Code)

The MATLAB code to do this, plane_exp2imp_3d.m

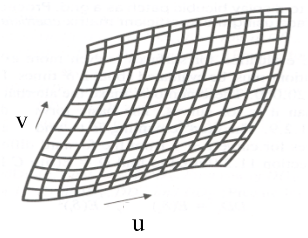
```
a = ( p2(2) - p1(2) ) * ( p3(3) - p1(3) ) ...
    - ( p2(3) - p1(3) ) * ( p3(2) - p1(2) );

b = ( p2(3) - p1(3) ) * ( p3(1) - p1(1) ) ...
    - ( p2(1) - p1(1) ) * ( p3(3) - p1(3) );

c = ( p2(1) - p1(1) ) * ( p3(2) - p1(2) ) ...
    - ( p2(2) - p1(2) ) * ( p3(1) - p1(1) );

d = - p2(1) * a - p2(2) * b - p2(3) * c;
```

Parametric Surfaces

The general form of a parametric surface is

$$\mathbf{r} = (x(u, v), y(u, v), z(u, v)).$$

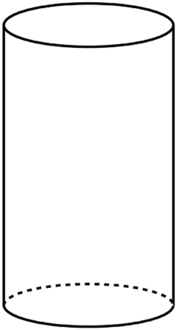
This is just like a parametric curve except we now have two parameters u and v that vary.

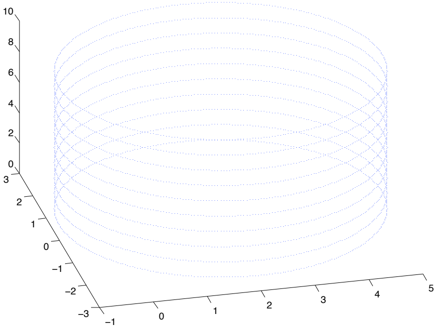


Parametric Surface: Cylinder

For example, a cylindrical may be represented in parametric form as

$$x = x_0 + r \cos u \qquad y = y_0 + r \sin u \qquad z = z_0 + v.$$





Parametric Surface: Cylinder (MATLAB Code)

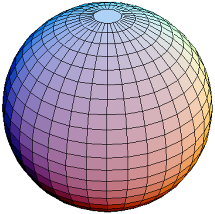
The MATLAB code to plot the cylinder figure is cyl_plot.m

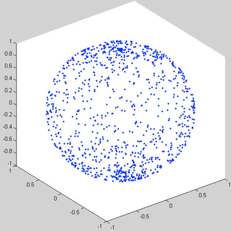
```
p0 = [2,0,0] % x_0, y_0, z_0
r = 3; %radius

n = 360;

hold on;
for v = 1:10
for u = 1:360
theta = ( 2.0 * pi * ( u - 1 ) ) / n;
x = p0(1) + r * cos(theta);
y = p0(2) + r * sin(theta);
z = p0(3) + v;

plot3(x,y,z);
end
end
```

Parametric Surface: Sphere

A sphere is represented in parametric form as

$$x = x_c + r\sin(u)\sin(v) \qquad y = y_c + r\cos(u)\sin(v) \qquad z = z_c + r\cos(v)$$





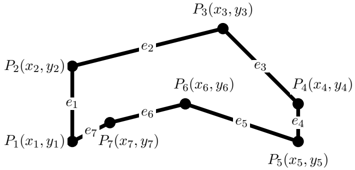
MATLAB code to produce a parametric sphere is at HyperSphere.m (see `help HyperSphere` for examples).

Piecewise Shape Models

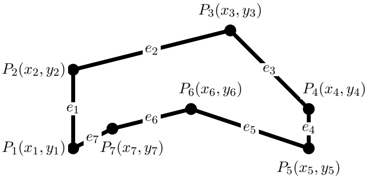
Polygons

A polygon is a 2D shape that is bounded by a closed path or circuit, composed of a finite sequence of straight line segments.

We can represent a polygon by a series of connected lines:



Polygons (Cont.)



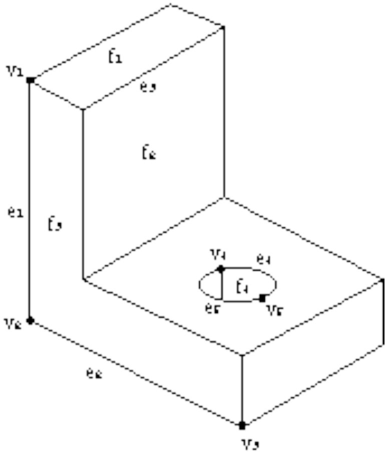
- Each line is defined by two vertices — the start and end points.
- We can define a data structure which stores a list of points (coordinate positions) and the edges define by indexes to two points:

  Points : $\{P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3), \ldots\}$
  Points define the **geometry** of the polygon.
  Edges : $\text{Edges}: \{e_1 = (P_1, P_2), e_2 = (P_2, P_3), \ldots$
  Edges define the **topology** of the polygon.

- If you traverse the polygon points in an ordered direction (clockwise) then the lines define a closed shape with the inside on the right of each line.

3D Objects: Boundary Representation

In 3D we need to represent the object's faces, edges and vertices and how they are joined together:



3D Objects: Boundary Representation

Topology — The topology of an object records the connectivity of the faces, edges and vertices. Thus,

- Each edge has a vertex at either end of it (e_1 is terminated by v_1 and v_2), and,
- each edge has two faces, one on either side of it (edge e_3 lies between faces f_1 and f_2).
- A face is either represented as an implicit surface or as a parametric surface
- Edges may be straight lines (like e_1), circular arcs (like e_4), and so on.

Geometry – This described the exact shape and position of each of the edges, faces and vertices. The geometry of a vertex is just its position in space as given by its (x, y, z) coordinates.

Summary

3D Geometric Computing:

- 3D Line Equations (Parametric)
- Implicit Surfaces (Planes, Spheres)
- Geometric Computation Related to 3D Lines and Planes
- Parametric Surfaces (Planes, Cylinders, Spheres)
- 3D Objects using Boundary Representation