CM2202: Scientific Computing and Multimedia Applications
General Maths: 5. Calculus: Integration

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Integration - Introduction

Integration is the **reverse** process of differentiation.

Integration

- Given $f'(x) = \frac{dy}{dx}$, find y or $f(x)$.

Example

- $f'(x) = \frac{d}{dx}(x^4) = 4x^3$
- Therefore the *reverse* process, the **integral** of $4x^3$, **has to be** x^4.

Integration — Notation

Integral Notation

We denote the integral of $f(x)$ as

$$\int f(x)dx$$

Within the integral, the expression $f(x)$ is called the **integrand**.

Constant of Integration

Consider the following derivatives:

$$\begin{aligned} \frac{d}{dx}(x^4) &= 4x^3 \\ \frac{d}{dx}(x^4+2) &= 4x^3 \\ \frac{d}{dx}(x^4-5) &= 4x^3 \end{aligned}$$

They all have the **same** derivative?

- The **derivative** of any constant is 0.

So what is the true value of $\int 4x^3 dx$?

- Have we lost trace of any derivative constants?

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Constant of Integration (2)

Constant of Integration

In **integration**, we have to **account** for the **reversing** of any derivative by adding in a constant, c say.

Given that $f'(x)$ is the derivative of $f(x)$ then, in general,

$$\int f'(x)dx = f(x) + c$$

Example

$$\int 4x^3 dx = x^4 + c$$

Integration of x^n

Suppose that $\frac{dy}{dx} = x^n$, for some constant n.

When we differentiate x^{n+1} ($\frac{d}{dx}(x^{n+1})$) we get $(n+1)x^n$:

$$\frac{d}{dx}(x^{n+1}) = (n+1)x^n$$

If we divide by $(n+1)$ we get:

$$\frac{1}{(n+1)}\frac{d}{dx}(x^{n+1}) = x^n$$

If we integrate both sides we get the formula we require.

$\int x^n dx$

$$\int x^n dx = \frac{x^{n+1}}{(n+1)} + c$$

Integration of x^n

Simple examples of applying the formula $\int x^n dx = \frac{x^{n+1}}{(n+1)} + c$

- $\int x \, dx = \frac{x^2}{2} + c$
- $\int x^2 dx = \frac{x^3}{3} + c$
- $\int x^9 dx = \frac{x^{10}}{10} + c$

Integration of Polynomial

Two important results:

Integral of two functions added together

Given **two** **integrable** functions $g(x)$ and $h(x)$ then:

$$\int (g(x) + h(x))dx = \int g(x)dx + \int h(x)dx$$

Integral of a function multiplied by a constant

If a is a **constant** and $f(x)$ is **integrable** then:

$$\int af(x)dx = a \int f(x)dx$$

Example: Integration of Polynomial

Evaluate

$$\int (8x^3 + 2x^2 + 3)dx$$

Solution:

$$\begin{aligned}
\int (\mathbf{8x^3} + \mathbf{2x^2} + \mathbf{3})\mathbf{dx} &= \int 8x^3 dx + \int 2x^2 dx + \int 3dx \\
&= 8\int x^3 dx + 2\int x^2 dx + 3\int 1dx \\
&= 8 \times \frac{x^4}{4} + 2 \times \frac{x^3}{3} + 3x + c \\
&= \mathbf{2x^4} + \frac{\mathbf{2}}{\mathbf{3}}\mathbf{x^3} + \mathbf{3x} + \mathbf{c}
\end{aligned}$$

Definite and Indefinite Integrals

Indefinite Integrals

The integrals so far have involved a constant of integration c

- These are **indefinite** integrals.

Definite Integrals

Sometimes we want to know the change of $y = \int f(x)dx$ between **two values** of x.

To do this we evaluate y at these **two values** of x and **subtract** the results.

- This is a **definite** integral.

Example: Definite Integral

Suppose $y' = 6x$ then the indefinite integral is
$y = \int 6x\, dx = 3x^2 + c$

To find the change of y between $x = 1$ and $x = 3$ we can put these values into the expression for y and subtract.

The change in y is

$$(3 \times 3^2 + c) - (3 \times 1^2 + c) = 27 + c - 3 - c = 24$$

This is the **definite integral**.

- A **definite integral** returns a **numeric value**
  - An **indefinite integral** returns a **function of** x
- **Notice** that the cs have **cancelled out**.

Definite Integral Notation

Notating the limits of a Definite Integral

- We show that the integral is to be **evaluated** between **two** values of x, x_1 and x_2,
  - We put these numbers at the **top** and **bottom** of the **integral sign**
  - We also put these numbers at the **top** and **bottom** of the **square brackets** delimiting the **integral result**:

$$\int_{x_1}^{x_2} f'(x)\,dx = [f(x)]_{x_1}^{x_2}$$

We also subtract the result of the **bottom** value, x_1, **from** the result of the **top** value, x_2.

Definite Integral Example Continued

So to continue our example from two slides ago and notate it **correctly**:

Total change in y is:

$$y = \int_1^3 6x\,dx = \left[3x^2\right]_1^3 = 3 \times 3^2 - 3 \times 1^2 = 24$$

No Constant in Definite Integration

Note: for **definite integrals** we **do not need to include** the **constant of integration**.

MATLAB: poly() Integrals

MATLAB lets you integrate poly() Polynomials:

polyint() and trapz()

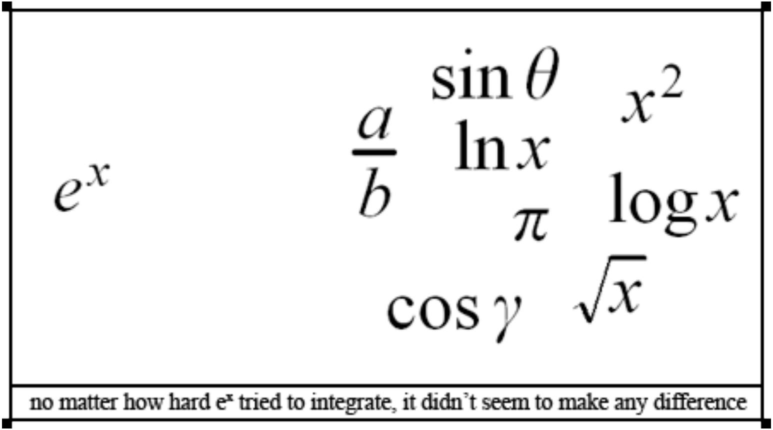
```
>> p = [1 0 0] % p = x^2

% Indefinite Integral
>> polyint(p)  % Ans = x^3/3

ans =  0.3333        0        0        0

% Definite Integral (via numerical integration)
>> trapz(-6:6,polyval(p,-6:6))

ans =  146
```

Integrating Other Functions

Well Known Integrals

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\log \mid \sec x \mid$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{a^2+x^2}}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$ (for $a > 0$)
e^{kx}	$\frac{e^{kx}}{k}$
$\frac{1}{x}$	$\log \mid x \mid$
$\log x$	$x \log x - x$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\log \mid \frac{a+x}{a-x} \mid$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\log \mid \frac{x-a}{x+a} \mid$
$\frac{1}{\sqrt{x^2 \pm a^2}}$	$\log \mid x + \sqrt{x^2 \pm a^2} \mid$
$\frac{x}{\sqrt{x^2 \pm a^2}}$	$\sqrt{x^2 \pm a^2}$
$\frac{x}{\sqrt{a^2-x^2}}$	$-\sqrt{a^2-x^2}$

Integration is important



In this module $\int e^{kx}\,dx$ is the most important integral - **see Fourier Theory**

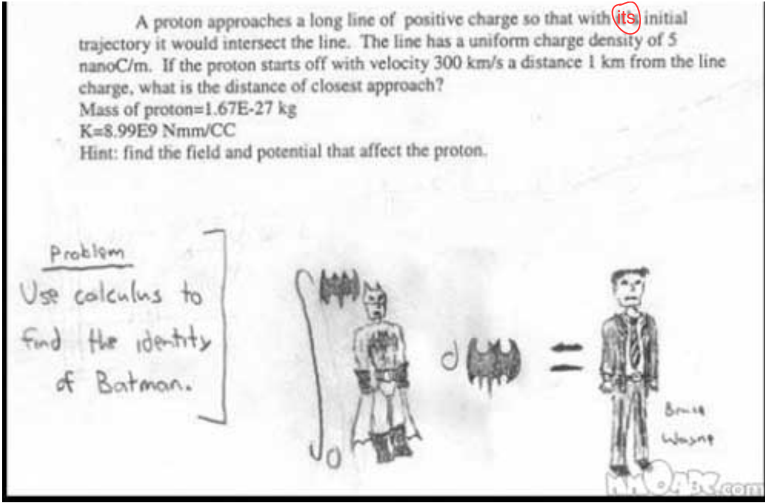
Cartoon Corollary (1)

$$\int \frac{1}{\text{cabin}}\, d\text{cabin} = \ln \text{cabin} + c$$

$$= \text{natural log cabin} + c$$

$$= \text{houseboat}$$

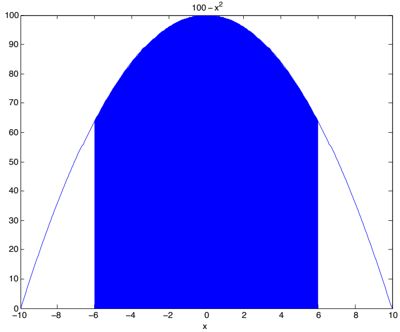
Cartoon Corollary (2)



Practical Application of Definite Integrals: Area Under a Curve

Area Under a Curve

For any given integrable curve, $f(x)$, the **definite integral** between two limits gives the **area under the graph between** these limits.

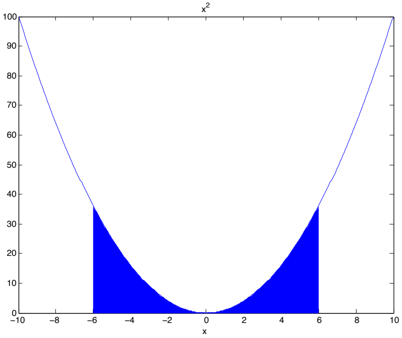


Example: Area Under a Curve

Area Under $f(x) = x^2$

Let limits be $x_1 = -6$ and $x_2 = 6$, then area under curve x^2 between $[x_1, x_2]$:

$$\begin{aligned} \int_{-6}^{6} \mathbf{x^2\,dx} &= \left[\frac{x^3}{3}\right]_{-6}^{6} \\ &= 72 - (-72) \\ &= \mathbf{144} \end{aligned}$$



MATALB Example: Area Under Graph (1)

MATLAB Code to solve and plot above example: integral_area.m

```
% declare function and plot it
syms x
plot_range = [-10,10];

f = x^2;
ezplot(f, plot_range);

hold on;

% define definite integral limits
int_limits = [-6  6];

% Integrate

intf = int(f)  % Indefinite Integral

% Definite Integral via Indefinite result
int_val_ind = int(f,int_limits(1),int_limits(2))
```

MATALB Example: Area Under Graph (2)

integral_area.m Cont.

```
% Idiot Check! Definite Integral via Indefinite result
subs(intf,int_limits(2)) - subs(intf,int_limits(1))
% not really a necessary bit of code as
%Def. Int is the way to do it!

% Set up plot
range = int_limits(1):0.1:int_limits(2);
y = subs(f,range); %sample values on curve

% Shade area below the curve
area(range,y, 'FaceColor',[0, 0, 1], 'LineStyle', 'none');
```

Note: `area()`- useful plot function to shade areas of given graph values.

MATLAB code for this at integral_area.m

Area Between Two Curves

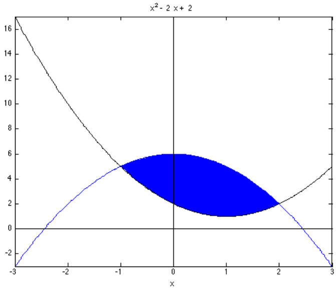
Area Between Two Curves

Suppose an area is enclosed between two curves $y = f(x)$ and $y = g(x)$.

Then we can find the **area**(s) **under** $y = f(x)$ and $y = g(x)$ by finding their integrals (**= area**) and taking their **difference**:

$$\text{Area} = \int_{x_1}^{x_2} f(x)\, dx - \int_{x_1}^{x_2} g(x)\, dx = \int_{x_1}^{x_2} (f(x) - g(x))\, dx$$

- The shaded area is the **difference** between these two integrals.



Example: Area Between Two Curves (1)

Area between $f(x) = -x^2 + 6$ and $g(x) = x^2 - 2x + 2$

Find the area of the region that is limited below the curve $f(x) = -x^2 + 6$ and above the curves $g(x) = x^2 - 2x + 2$.

First we need to find the **points of intersection** of the curves, setting $f(x) = g(x)$

$$\begin{aligned} -x^2 + 6 &= x^2 - 2x + 2 \\ 2x^2 - 2x - 4 &= 0 \\ (x+1)(x-2) &= 0 \\ \text{gives:} & \\ \mathbf{x} &= -\mathbf{1} \text{ and } \mathbf{2} \end{aligned}$$

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Example: Area Between Two Curves (2)

Area between $f(x) = -x^2 + 6$ and $g(x) = x^2 - 2x + 2$

Given points of intersection $\mathbf{x = -1}$ and $\mathbf{2}$, we can now integrate $f(x) - g(x)$ between these limits to find the area between these curves:

$$\begin{aligned}
\textbf{Area} &= \int_{-1}^{2} ((-x^2 + 6) - (x^2 - 2x + 2))\, dx \\
&= \int_{-1}^{2} (-2x^2 + 2x + 4)\, dx \\
&= \left[-\frac{2}{3}x^3 + x^2 + 4x \right]_{-1}^{2} \\
&= \mathbf{9}
\end{aligned}$$

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Example: Area Between Two Curves (1)

MATLAB Code: integral_area_between.m

```
% declare functions
syms x
plot_range = [-3,3];

f1 = -x^2 + 6;
f2 = x^2 - 2*x + 2;

% Find Points of Intersection of curves
roots_intersect = sort(double(solve (f1 - f2)));

% Definite Integral
area_intersect =
    int(f1 - f2,roots_intersect(1),roots_intersect(2))
```

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Example: Area Between Two Curves (1)

integral_area_between.m Cont.

```
% Set up plot
range = roots_intersect(1):0.1:roots_intersect(2);
y = subs(f1,range); %sample values on curve f1

% Shade Area f1
area(range,y, 'FaceColor',[0, 0, 1], 'LineStyle', 'none')
hold on;

% 'rub out' area above the curve
y = subs(f2,range); %sample values on curve f2
area(range,y, 'FaceColor',[1, 1, 1], 'LineStyle', 'none')

% Plot funtions over plot range
ezplot(f1, plot_range);
hf2 = ezplot(f2, plot_range);
set(hf2,'Color','black');
```

MATLAB code for this at integral_area_between.m

Area Under x-axis

Area Under x-axis

If the curve of $y = f(x)$ lies **below** the x-axis, we find that the integral of $f(x)$ is **negative**:

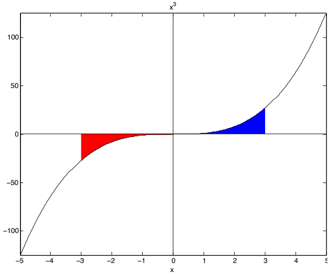
- So the area below the x-axis is counted as negative in integral evaluation.

If we want to find the **actual** area, we **must** take **account of this**:

- we have to split the integration in separate parts, **e.g.**:

$$\int_{x_1}^{x_2} f(x)\,dx = \left|\int_{x_1}^{x_0} f(x)\,dx\right| + \left|\int_{x_0}^{x_2} f(x)\,dx\right|$$

where x_0 is point where the curve intersects the x-axis between $[x_1, x_2]$



Note: It is **possible** there are **more than 1 intersections** within a limit $[x_1, x_2]$.

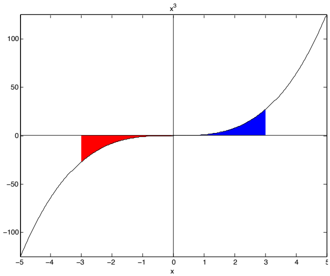
Example: Area Under x-axis: Area of x^3

$y = x^3$ is an **odd** function.

- It **crosses** the x-axis at $x = 0$ — root(s) at $x = 0$

Now $\int_{-n}^{n} x^3 \, dx$ will always be zero for any n - since x^3 is odd.

- Any $\int_{-n}^{0} x^3 \, dx$ will be negative



To calculate the area (between $x = [-3, 3]$, say) we need to do:

$$\left|\int_{-3}^{0} x^3 \, dx\right| + \left|\int_{0}^{3} x^3 \, dx\right| = \left|\left[\frac{x^4}{4}\right]_{-3}^{0}\right| + \left|\left[\frac{x^4}{4}\right]_{0}^{3}\right|$$

$$= \left|-\frac{(-3)^4}{4}\right| + \left|\frac{(3)^4}{4}\right|$$

$$= \frac{81}{4} + \frac{81}{4} = \mathbf{\frac{81}{2}}$$

MATLAB Example: Area Under x-axis (1)

MATLAB Code: integral_area_under.m

```
% declare function
syms x
plot_range = [-5,5];
int_limits =  [-3,3];

f = x^3;

% Definite Integral via Indefinite result
int_area = int(f,int_limits(1),int_limits(2))

% WRONG (for Area)!  Answer is ZERO
% For x^3 need to split at x = 0
% Exercise: Write a general function to work this point out.

% Do Area Properly
int_area = abs(int(f,int_limits(1),0)) + abs(int(f,0,int_limits(2)))
```

MATLAB Example: Area Under x-axis (2)

MATLAB Code: integral_area_under.m

```
% Set up plot
range = int_limits(1):0.1:0;
y = subs(f,range); %sample values on curve

% Shade area below the curve
area(range,y, 'FaceColor',[1, 0, 0], 'LineStyle', 'none');
hold on;

range = 0:0.1:int_limits(2);
y = subs(f,range); %sample values on curve

% Shade area below the curve
area(range,y, 'FaceColor',[0, 0, 1], 'LineStyle', 'none');

%plot function
hf =ezplot(f, plot_range);

set(hf,'Color','black');
```

MATLAB code for this at integral_area_under.m

Compound Integrals: Integration by Parts

The product rule tells us how to **differentiate** the product of two functions.

- There is no *infallible* rule for integrating a product but one of the techniques available is **Integration by Parts**.

Integration by Parts

$$\int \frac{du}{dx} v \, dx = uv - \int u \frac{dv}{dx} \, dx$$

or

$$\int u' v \, dx = uv - \int u v' \, dx$$

- It may appear that the **right hand side** of the above eqn. is more complicated than the original product on the **left hand side**.
  - However, by careful choice of v we can ensure that $\int u \frac{dv}{dx} \, dx$ or uv' is **easier** to **integrate** than $\int \frac{du}{dx} v \, dx$ or vu'.
  - The **main rule** to selecting v is let v be the simpler function of the compound.

Mathematical Proof for Integration by Parts

Integration as the reverse of differentiation:

Recall the differentiation of a product uv where u and v are both functions of x we have:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Rewrite as:

$$v\frac{du}{dx} = \frac{d}{dx}(uv) - u\frac{dv}{dx}$$

Integrate both sides to give:

$$\int \mathbf{u'v\,dx} = \mathbf{uv} - \int \mathbf{uv'\,dx}$$

Example: Integration by Parts

Find $\int x\cos(x)$

We know how to integrate and differentiate both x and $\cos(x)$
Following above rules

- Let $v = x \rightarrow \frac{dv}{dx} = 1$
  - Always let v be the simpler function, $v = x$ in this case.
- Let $\frac{du}{dx} = \cos(x) \rightarrow \mathbf{u} = \sin(\mathbf{x})$

Apply formula:

$$\int \mathbf{x}\cos(\mathbf{x}) = x\sin(x) - \int \sin(x) = \mathbf{x}\sin(\mathbf{x}) + \mathbf{cos}(\mathbf{x})$$

Note: Setting $v = \cos(x)$ and $\frac{du}{dx} = x$ gets rather messy!

Integration Via Substitution

Substitution — a very powerful general mathematical trick — can assist Integration.

Integral Substitutions

If $x = g(u)$ then

$$\int f(x)dx = \int f(g(u))\frac{dx}{du}du$$

or, if $u = \phi(x)$ then

$$\int f(\phi(x))dx = \int f(u)\frac{dx}{du}du$$

Example: Integration Via Substitution (1)

Integration Via Substitution: $\int f(x)dx = \int f(g(u))\frac{dx}{du}du$

Integrate $\int (x+4)^5\, dx$

Let $u = x + 4$

Now $\int u^5 = \frac{u^6}{6} + c$ and $\frac{du}{dx} = 1 \rightarrow du = dx$ so we have:

$$\int (x+4)^5\, dx = \frac{(x+4)^6}{6} + c$$

Example: Integration Via Substitution (2)

Integration Via Substitution: $\int f(x)dx = \int f(g(u))\frac{dx}{du}du$

Integrate $\int 2x\sqrt{1+x^2}\,dx$ Let $u = 1 + x^2$ Now

$\frac{du}{dx} = 2x \rightarrow \frac{dx}{du} = \frac{1}{2x}$ so we have:

$$\begin{aligned}
\int 2x\sqrt{1+x^2}\,dx &= \int \sqrt{u}\,du \\
&= \int u^{\frac{1}{2}}\,du \\
&= \frac{2}{3}u^{\frac{3}{2}} + c \\
&= \mathbf{\frac{2}{3}(1+x^2)^{\frac{3}{2}} + c}
\end{aligned}$$

Double Integrals

We will shortly see a few examples of a **function of two variables** and need to **integrate** such a function:

- An image (2D stucture) is an example of a function of two variables.
- Fourier Theory applied to Image processing requires the integration of a Image function of two variables

Notation: Integration of a function of two variables

If a function, $f(x, y)$, is a function of **two** variables, x and y, how do we integrate:

$$\int\int f(x,y)\,dx\,dy?$$

Double Integration

This is no more difficult than single integration.

How to do Double Integration

Split the integration into two steps:

- **The Inner Integral**: Integrate the inner integral with **respect to** x:

$$\int \left[\int \mathbf{f(x, y)\, dx} \right] dy$$

  I.e **Inner Integral** $= \int \mathbf{f(x, y)\, dx}$

  Regard y as a **constant** here.

  - We could also do the inner integral with respect to $y \rightarrow x$ and y are **interchangeable**.

- **The Outer Integral**: Integrate the outer integral on the **results** of the **inner integral** with **respect to** y treating x as a **constant**:

$$\int \textbf{Inner Integral dy}$$

Example: Double Integration (1)

Evaluate $\int \int x^2 + y^2 \, dx \, dy$

Evaluate $\int \int x^2 + y^2 \, dx \, dy$:

- **Inner integral** is $\int x^2 + y^2 \, dx = \mathbf{\frac{x^3}{3}} + \mathbf{x}y^2$
- Outer integral is $\int \frac{x^3}{3} + xy^2 \, dy = \frac{x^3}{3}\mathbf{y} + x\mathbf{\frac{y^3}{3}}$

So

$$\int \int x^2 + y^2 \, dx \, dy = \frac{x^3}{3}y + x\frac{y^3}{3} + c$$

Example: Double Integration (2)

Evaluate $\int \int e^{kx+ly}\ dx\ dy$

Evaluate $\int \int e^{kx+ly}\ dx\ dy, k, l \in \mathbb{R}$
Now $\int \int e^{kx+ly}\ dx\ dy = \int \int e^{kx} e^{ly}\ dx\ dy$

- **Inner integral** is $\int e^{kx} e^{ly}\ dx = \frac{\mathbf{e^{kx}}}{\mathbf{k}} e^{ly}$
- Outer integral is $\int \frac{e^{kx}}{k} e^{ly}\ dy = \frac{e^{kx}}{k} \frac{\mathbf{e^{ly}}}{\mathbf{l}}$

So

$$\int \int \mathbf{e^{kx+ly}\ dx\ dy} = \frac{e^{kx}}{k} \frac{e^{ly}}{l} = \frac{\mathbf{1}}{\mathbf{kl}} \mathbf{e^{kx+ly}} + c$$

MATLAB Integration Examples (1)

MATLAB Integration by Parts and Substitution, Integration_egs.m

```
>>syms x

% MATLAB can do Integration by Parts
>>f = x*cos(x);
>>int_f = int(f)
>> int_f = cos(x) + x*sin(x)

% MATLAB can do Integration by Substitution
>>f = (x+4)^5;
>>int_f = int(f)
>>int_f =  (x + 4)^6/6


>>f = 2*x*sqrt(1 + x^2);
>>int_f =  int(f)
>>int_f = (2*(x^2 + 1)^(3/2))/3
```

MATLAB code for this at Integration_egs.m

MATLAB Integration Examples (2)

MATLAB: Double integrals, Integration_egs.m

```
% Double Integration
>>syms x
>>syms y

>>f = exp(2*x+3*y);
>>int_f = int(int(f,x),y)
>>int_f =  exp(2*x + 3*y)/6
```

MATLAB code for this at Integration_egs.m