Chapter 4: Linear Algebra, Vectors and Matrices

Vectors and Matrices are a staple data structure in many areas of Computer Science.

Computer Graphics is one prime example — here linear algebra permeates almost every area.

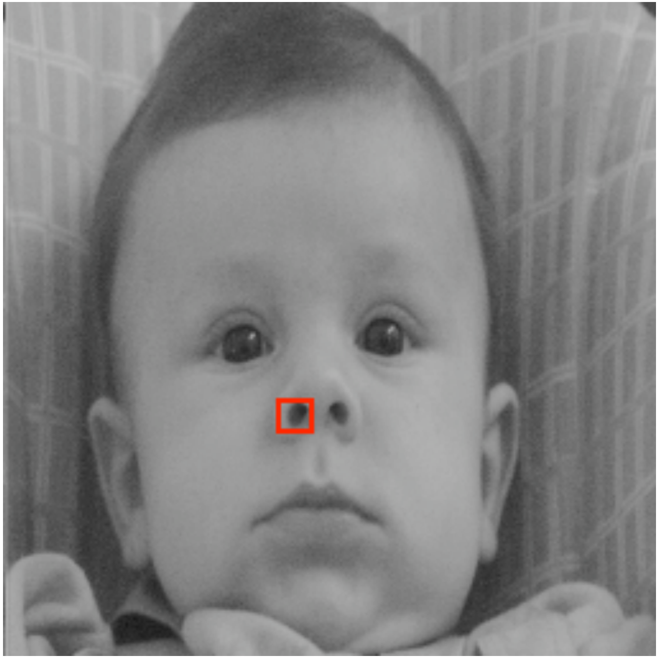
We will use some simple examples from Computer Graphics to visualise some simple aspects of Linear Algebra, Vectors and Matrices.

We will use other examples as appropriate.

Selected Examples of Use in Computer Science

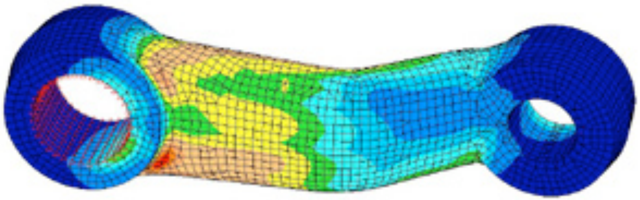
- Basic Linear Algebra — solutions of equations needed in almost every scientific discipline
- Vectors and Matrices — fundamental data structures in computer science e.g. *Arrays, Linked Lists*
- Numerical Analysis — scientific computing and practical computational mathematics
- Computer Graphics: Transformations, moving object around the screen, 3D deformations . . .
- Image Processing/Computer Vision: Images = matrices, Tracking objects, Object Recognition, Camera Calibration . . .
- Data Compression: JPEG/MPEG, Image/Video/Audio Compression, Vector Quantisation

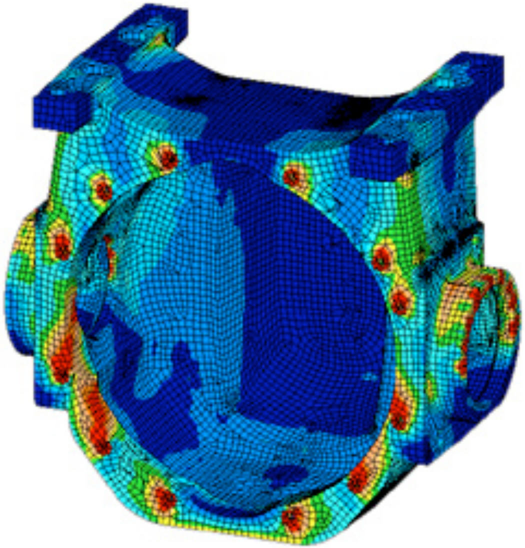
Matrices Example: Image Representation



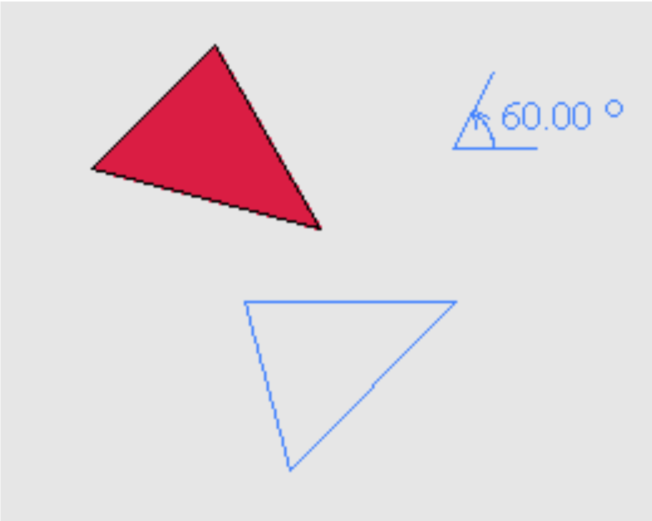
99	71	61	51	49	40	35	53	86	99
93	74	53	56	48	46	48	72	85	102
101	69	57	53	54	52	64	82	88	101
107	82	64	63	59	60	81	90	93	100
114	93	76	69	72	85	94	99	95	99
117	108	94	92	97	101	100	108	105	99
116	114	109	106	105	108	108	102	107	110
115	113	109	114	111	111	113	108	111	115
110	113	111	109	106	108	110	115	120	122
103	107	106	108	109	114	120	124	124	132

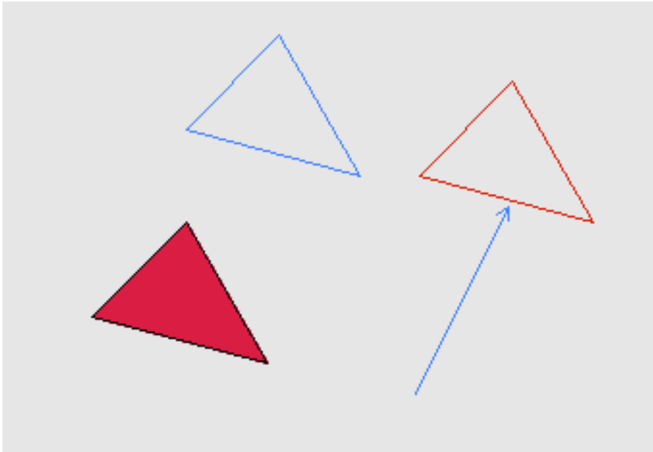
Algebra/Graphs Example: Finite Element Modelling





Matrices Example: Computer Graphics Transformations

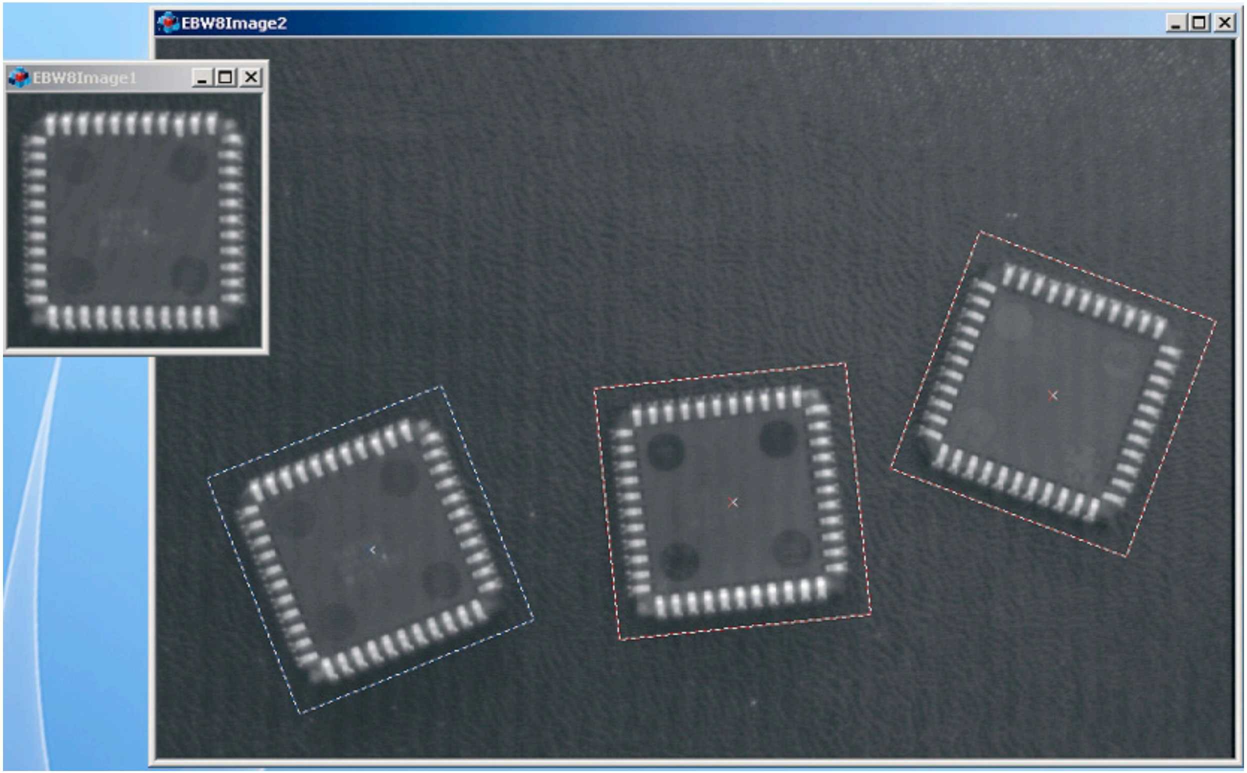




$$\begin{bmatrix} X_{rotated} \\ Y_{rotated} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{translated} \\ Y_{translated} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & D_x \\ 0 & 1 & D_y \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Matrices Example: Object Registration/Matching



Matrices Example: Image Warping (Transformation)

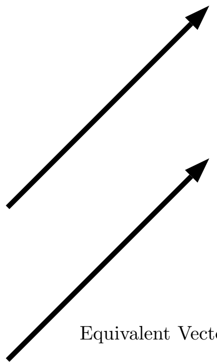
Matrices/Vector Example: Image Compression

2- 3- and n-dimensional vectors

Vector basics

Definition 4.1 (2-dimensional vectors).

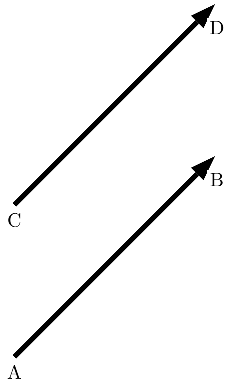
We define two-dimensional vectors as directed arrows in the plane. A vector is determined by the length and the direction of the arrow. Two vectors are called equivalent if they have the same length and direction.



Equivalent Vectors

Back
Close

Example 4.1 (2-dimensional vectors).

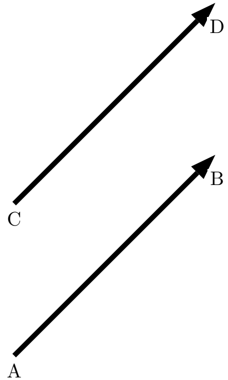


Vectors can be determined by *two points*. E.g. the vectors $\vec{AB}$ and $\vec{CD}$.

In the above example:

- A is called the *tail* of the vector $\vec{AB}$.
- B is called the *head* of the vector $\vec{AB}$.

Definition 4.2 (Equivalence of vectors).



Although $\vec{AB}$ and $\vec{CD}$ have different heads and tails, they are equivalent

- *We* **distinguish** *vectors* **only** *by their* direction *and* length.

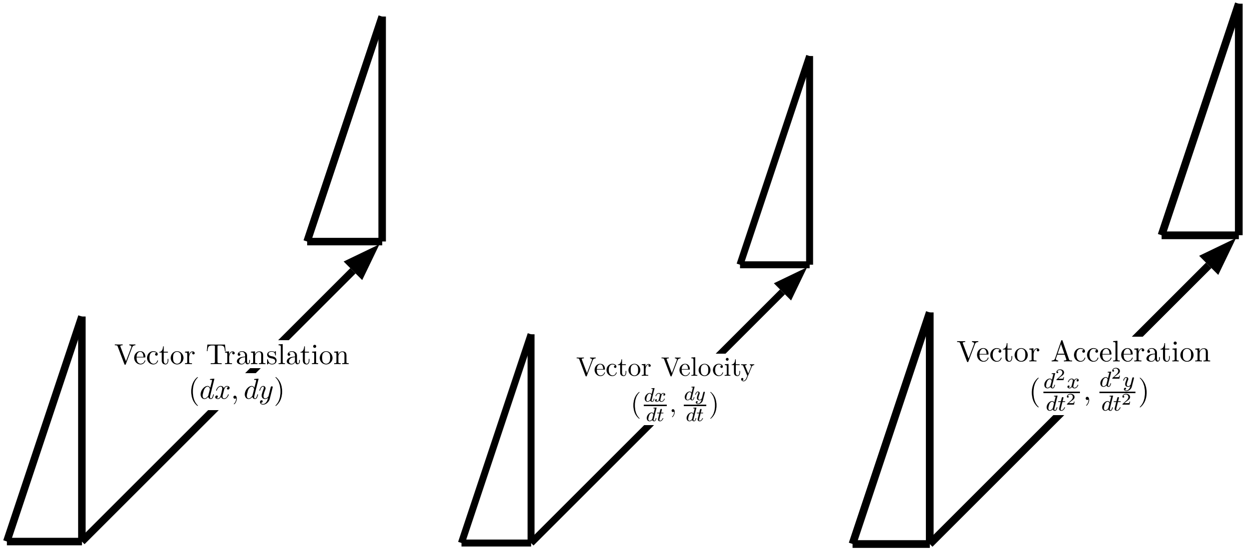
Thus we treat equivalent vectors as equal.

E.g.

$$\vec{AB} = \vec{CD}$$

Example 4.2 (Some Real World Examples of Vectors).

Vectors can be use to represent translation (motion), velocity, acceleration:



Definition 4.3 (Vector Representation/Notation).

Vectors can be defined in a variety of ways:

- As we have seen already by **two points**.

  In which case we use the notation $\vec{AB}$.

  Alternative notations (which we do not use but you may see in some books) are $\underline{AB}$ or $\underset{\sim}{AB}$ or $\tilde{AB}$

- A vector may also be defined as a **line** whose tail is the origin and whose head coordinates are given as a (x, y) pair (and similar for higher dimensions — **more soon**.

  In this case we use the notation $\mathbf{a} = (x, y)$ or $\vec{a} = (x, y)$

Some Other Vector Notations You May See

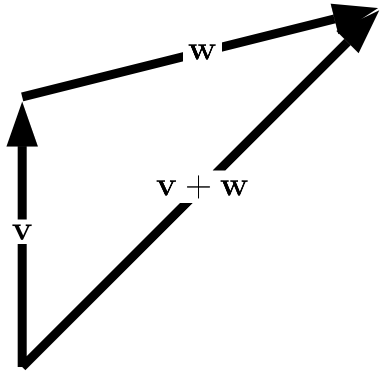
An alternative notations $\langle x, y \rangle, \underline{a}, \tilde{a}$ or, even, $\underset{\sim}{a}$.

Standard Vector/Matrix Notation Conventions

Note: It is standard notation to use a *lower case* letter for vectors (along with **bold**, $\vec{\text{vector}}$ *etc.* of course)

Bold upper case letters are reserved for matrices — **more later**.

Definition 4.4 (The triangle law).



We add two vectors $\mathbf{v}$ *and* $\mathbf{w}$ *in the following way.*

We arrange $\mathbf{w}$ *such that its tail coincides with the head of* $\mathbf{v}$.

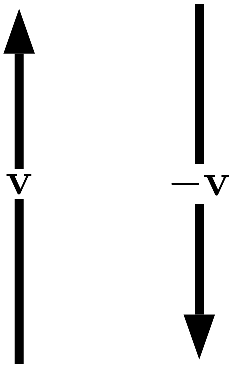
$\mathbf{u} = \mathbf{v} + \mathbf{w}$ *is then defined as the vector with the tail of* $\mathbf{v}$ *and the head of the newly arranged vector* $\mathbf{w}$*:*

Definition 4.5 (**0** and opposite vectors)**.**

We define $\mathbf{0}$ *as the vector with length* 0.

If $\mathbf{v}$ *is not* $\mathbf{0}$, *then we define* $-\mathbf{v}$ *as the vector with the same length and the opposite direction as* $\mathbf{v}$.

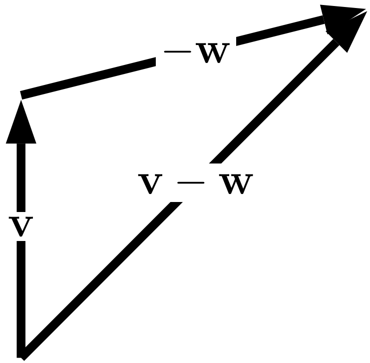
We see that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.



Definition 4.6 (The difference of two vectors).

The difference of two vectors $\mathbf{v}$ *and* $\mathbf{w}$ *is defined as*

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$



Definition 4.7 (Scalar multiplication)**.**

Let $\mathbf{v}$ *be a vector and* k *a real number.*

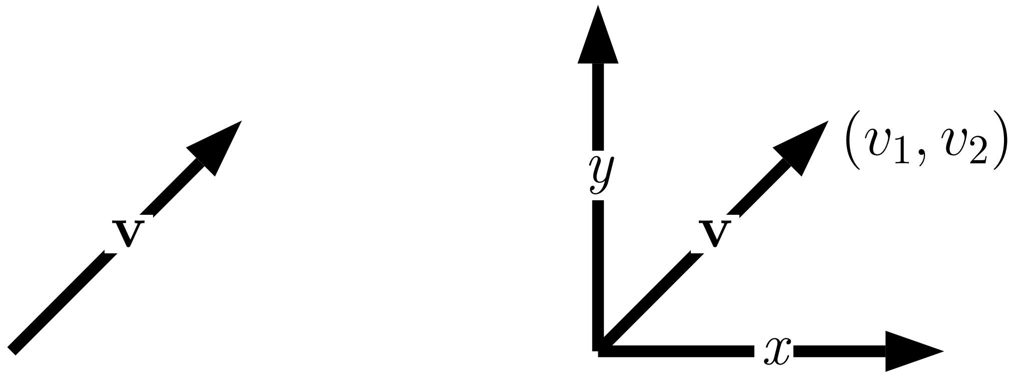
The vector $k\mathbf{v}$ *is defined as the vector with the same direction as* $\mathbf{v}$ *if* k *is positive and the opposite direction if* k *is negative.*

The length of $k\mathbf{v}$ *is* $|k| \times$ *the length of* $\mathbf{v}$.

Vectors in coordinate systems

We can simplify the analysis of vectors by introducing coordinate systems.

We consider the standard coordinate system in the $x - y$ (2D $\mathbb{R}^2$) plane:



If $\mathbf{v}$ is a two-dimensional vector we can always arrange it such that its tail coincides with the origin.

The coordinates (v_1, v_2) of its head uniquely identify $\mathbf{v}$ and are called the components of $\mathbf{v}$.

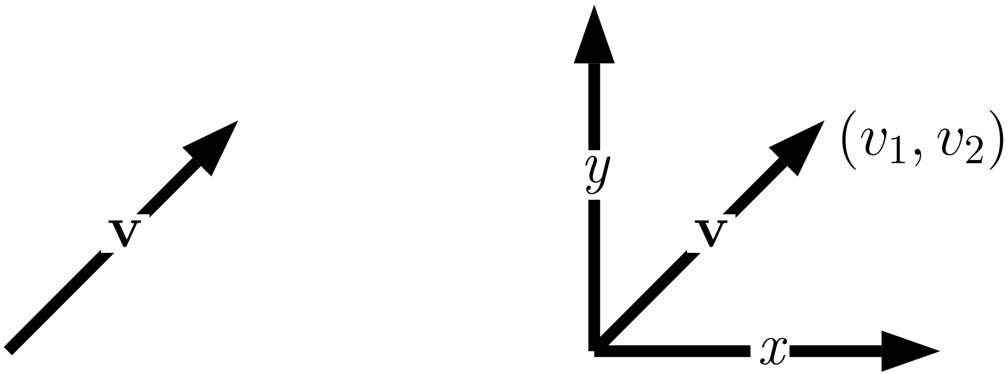
Vector Definition in $\mathbb{R}^2$

Since the coordinates of the head determine any vector uniquely, we make the following definition:

Definition 4.8 (Vectors in the space $\mathbb{R}^2$)**.**

We identify the space of two-dimensional vectors with

$$\mathbb{R}^2 = \{(x, y) | \; x \in \mathbb{R}, \; y \in \mathbb{R}\}$$

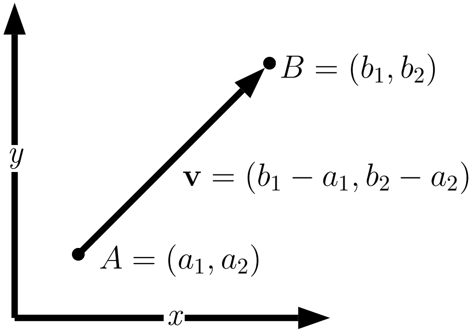


We now write the notation $\mathbf{v}$ for the vector (v_1, v_2).

Calculating Vectors from 2 Points in $\mathbb{R}^2$

If a vector $\mathbf{v}$ is defined by two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ we can get the components of $\mathbf{v}$ by the simple calculation:

$$\mathbf{v} = (b_1 - a_1, b_2 - a_2) \quad \text{Head - Tail}$$



Thus two vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ are **equivalent** if

$v_1 = w_1$ **and** $v_2 = w_2$.

Vectors in Higher Dimensional Spaces

Since we identified the space of two-dimensional vectors with the space of all ordered 2-tuples we can define higher dimensional vector spaces in the same way.

Definition 4.9 (Vectors in the spaces $\mathbb{R}^3$ and $\mathbb{R}^n$)**.**

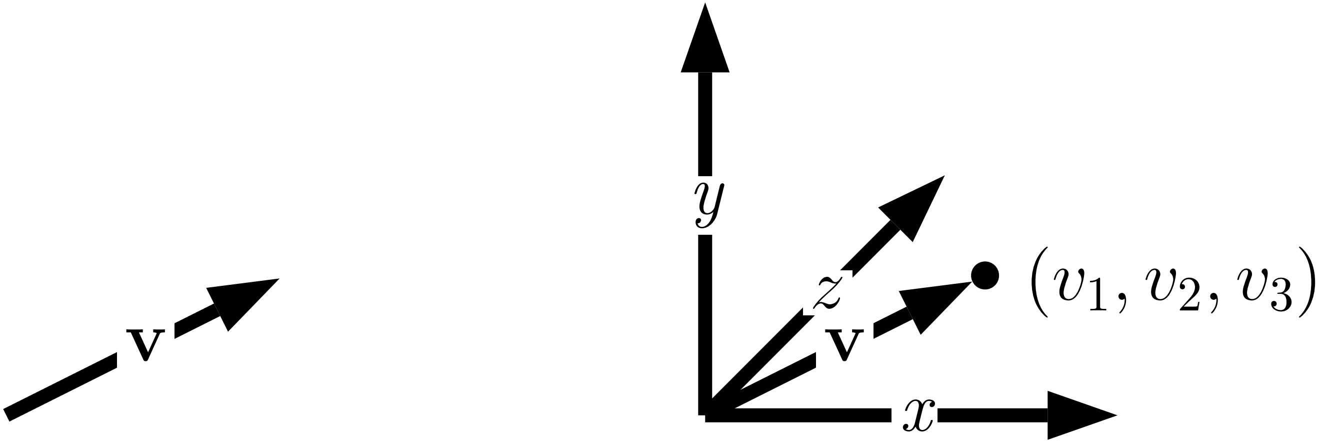
We define the space of three-dimensional vectors as

$$\mathbb{R}^3 = \{(x, y, z) |\ x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}\}$$

Let n be a positive integer. We define the space of n-dimensional vectors as

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) |\ x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ \ldots,\ x_n \in \mathbb{R}\}$$

Example 4.3 (Standard coordinate system in $\mathbb{R}^3$).



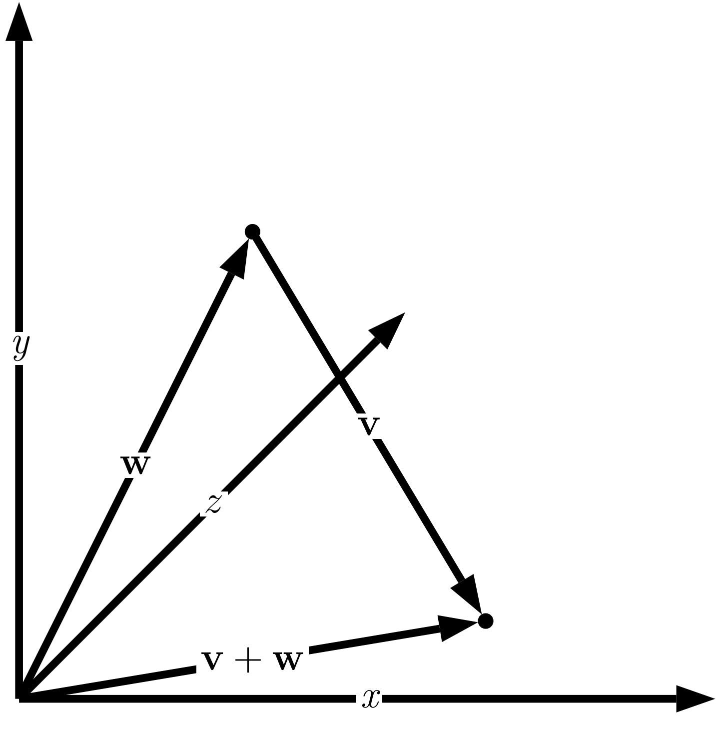
Definition 4.10 (Vector Addition, Subtraction and Scalar Multiplication in $\mathbb{R}^n$).

Let $\mathbf{v}$ *and* $\mathbf{w}$ *be two vectors in* $\mathbb{R}^n$ *and* k *a real number. The following rules are well-defined:*

- $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$.
- $\mathbf{v} - \mathbf{w} = (v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n)$
- $k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$.

These rules coincide with the geometrical interpretation for two-dimensional vectors (see Definitions 4.4, 4.6 and 4.7).

Example 4.4 (Vector Addition, Subtraction and Scalar Multiplication in $\mathbb{R}^n$).



Definition 4.11 (Scalar product).
Given two vectors v *and* w *in* $\mathbb{R}^n$ *with components* $(v_1, v_2, \ldots, v_n)$ *and* $(w_1, w_2, \ldots, w_n)$. *We define the* scalar product *of* $\mathbf{v}$ *and* $\mathbf{w}$ *as*

$$\mathbf{v}.\mathbf{w} \text{ or } \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i w_i$$

Note what the scalar product does:

It takes two vectors and assigns them a real number.

Problem 4.1 (Scalar product).

Work out the scalar product of vectors $\mathbf{v} = (1, 2)$ *and* $\mathbf{w} = (2, 3)$

Note the notations $\mathbf{v}.\mathbf{w}$ and $\langle \mathbf{v}, \mathbf{w} \rangle$ are equivalent.
We use the $\mathbf{v}.\mathbf{w}$ notation.

Theorem 4.12 (Scalar product properties).

The scalar product has the following properties.

- $\mathbf{v}.\mathbf{w} \geq 0$, *for all* $v \in \mathbb{R}^n$ *and* $\mathbf{v}.\mathbf{w} = 0 \iff v = 0$.
- $\mathbf{v}.\mathbf{w} = \mathbf{w}.\mathbf{v}$, *for all* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
- $(\mathbf{v} + \mathbf{u}).\mathbf{w} = \mathbf{v}.\mathbf{w} + \mathbf{u}.\mathbf{w}$, *for all* $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^n$.
- $(k\mathbf{v}).\mathbf{w} = k(\mathbf{v}.\mathbf{w})$ *for all* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $k \in \mathbb{R}$.

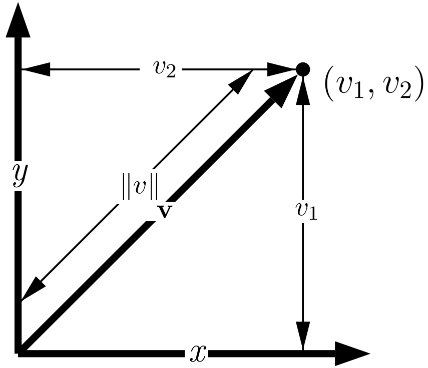
Definition 4.13 (Euclidean norm of a vector).

For a vector $\mathbf{v} \in \mathbb{R}^n$ *we define its norm as*

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}.\mathbf{v}}$$

This norm is called the euclidean norm of the vector $\mathbf{v}$.

The euclidean norm of a vector coincides with the length of the vector in $\mathbb{R}^2$ and $\mathbb{R}^3$.



By Pythagoras' Theorem, $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{\mathbf{v}.\mathbf{v}}$

Theorem 4.14 (Cauchy-Schwarz inequality).

Let v and w be vectors in $\mathbb{R}^n$

Then they satisfy the Cauchy-Schwarz inequality

$$\mathbf{v}.\mathbf{w} \leq \|\mathbf{v}\|\|\mathbf{w}\|.$$

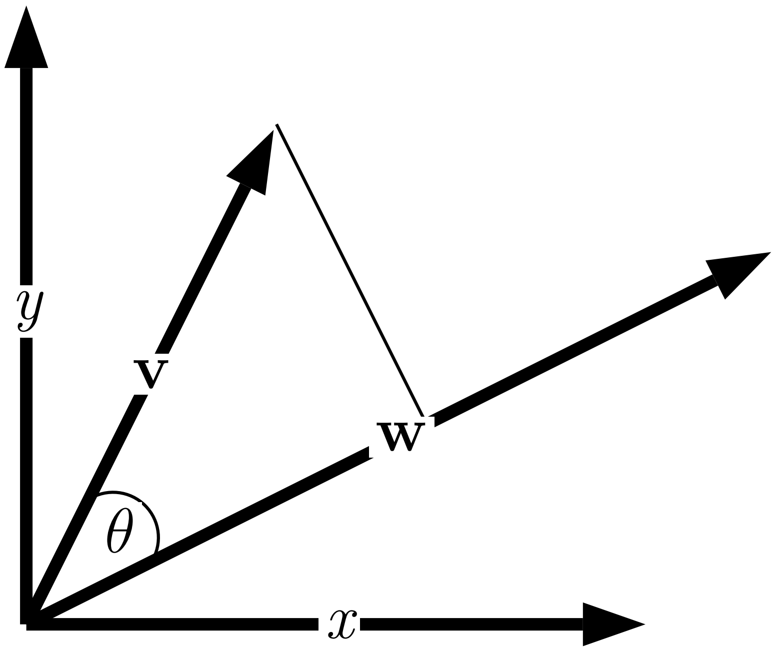
Theorem 4.15 (Angle Between Two Vectors).

If $n = 2, 3$ we even have the relation

$$\mathbf{v}.\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$$

We call θ the angle between v and w.

Geometric Visualisation of Angle Between Two Vectors in $\mathbb{R}^2$



Example 4.5 (Orthogonal vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$).

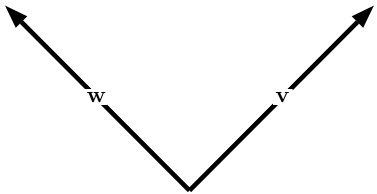
Let $\mathbf{v} = (v_1, v_2)$ *and* $\mathbf{w} = (w_1, w_2)$ *be two vectors in* $\mathbb{R}^2$.

We call $\mathbf{v}$ *and* $\mathbf{w}$ **orthogonal** *if the angle between them is* $90°$.

Since $\cos(\theta) = 0$ **if and only if** $\theta = 90°$ *for* $\theta \in [0, 180°]$ *we can conclude that orthogonal vectors are characterized by the relation*

$$\mathbf{v}.\mathbf{w} = 0.$$

This expression is also meaningful in $\mathbb{R}^n$ *and we say that two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^n$ *are* **orthogonal** *,* **if** *their* scalar product *is* **zero**.



The Vector Cross Product

Besides the scalar product that maps two vectors from $\mathbb{R}^n$ to $\mathbb{R}$ we also need a product that maps two vectors from $\mathbb{R}^n$ to a vector in $\mathbb{R}^n$.

Definition 4.16 (The vector cross product in $\mathbb{R}^2$).

We define the **vector cross product** *of* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ *as a mapping* $\times : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ *with*

$$\mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1$$

The vector product in $\mathbb{R}^2$ *is anti-symmetric, i.e.*

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$$

Definition 4.17 (The vector cross product in $\mathbb{R}^3$).

We define the vector cross product of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *as a mapping* $\times : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ *with*

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}.$$

The vector product in $\mathbb{R}^3$ *is also anti-symmetric, i.e.*

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$$

The vector cross product has very useful properties, especially:

- for finding orthogonal vectors in $\mathbb{R}^3$.
- for area and volume calculations in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Example 4.6 (Vector cross product: Orthogonal Vectors).

Work out the vector cross product of the vectors $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *and* $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

It is easy to show that:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

Example 4.7 (Vector cross product: volume calculations).

Let $n = 2$.

If $\mathbf{v} = (v_1, v_2)$ *then the vector* $\mathbf{v}^{perp} = (-v_2, v_1)$ *is orthogonal to* v.

It follows from the definition of the vector and scalar product, that

$$\mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1 = -v_2 w_1 + v_1 w_2 = \mathbf{v}^{perp} . \mathbf{w}$$

This expression is 0 *if* $\mathbf{w}$ *and* $\mathbf{v}^{perp}$ *are* **orthogonal**.

However this means that $\mathbf{v}$ *and* $\mathbf{w}$ *are* **parallel**.

Theorem 4.18 (Parallel vectors in $\mathbb{R}^2$).

We call two vectors $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^2$ parallel *if we have* $\mathbf{v} \times \mathbf{w} = 0$.

We even have

$$\mathbf{v} \times \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \sin(\theta)$$

where θ *is the angle between* $\mathbf{v}$ *and* $\mathbf{w}$ *counted positive counter-clockwise and negative clockwise starting from* $\mathbf{v}$.

Generalisation of sinusoidal relation

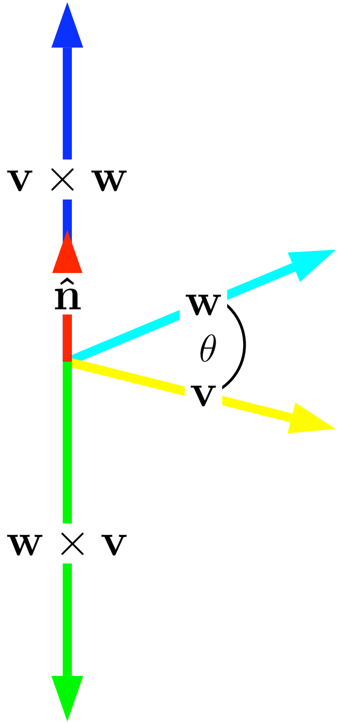
In general for any dimension it can be stated that:

$$\|\mathbf{v} \times \mathbf{w}| = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)$$

We also have:

$$\mathbf{v} \times \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)\hat{\mathbf{n}}$$

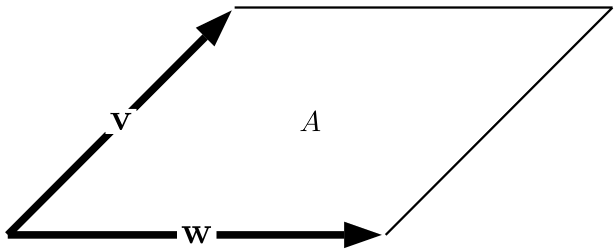
where $\hat{\mathbf{n}}$ *is a unit vector (of length 1) perpendicular to both* $\mathbf{v}$ *and* $\mathbf{w}$



Back to our Volume Calculation

If a parallelogram is spanned by $\mathbf{v}$ *and* $\mathbf{w}$ *then its area* A *is given by*

$$A = |\mathbf{v} \times \mathbf{w}|.$$



Volume Calculation in $\mathbb{R}^3$

Now let $n = 3$ *and let* $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^3$.

We have similar relationships as in the case $n = 2$.

One can show that

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\||\sin\theta|$$

where θ *is the angle between* $\mathbf{v}$ *and* $\mathbf{w}$.

In particular $\mathbf{v}$ *and* $\mathbf{w}$ *are* **parallel only if**

$$\mathbf{v} \times \mathbf{w} = \mathbf{0} \quad \mathbf{0} \textit{ is the zero vector in } \mathbb{R}^3.$$

Volume Calculation in $\mathbb{R}^3$: Scalar product/Cross Product

Now consider the expression: $\mathbf{v}.(\mathbf{v} \times \mathbf{w})$

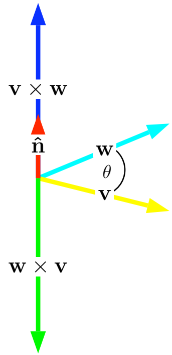
It holds that

$$\mathbf{v}.(\mathbf{v} \times \mathbf{w}) = v_1(v_2w_3 - v_3w_2) + v_2(v_3w_1 - v_1w_3) + v_3(v_1w_2 - v_2w_1) = 0$$

Similarly we can show

$$\mathbf{w}.(\mathbf{v} \times \mathbf{w}) = 0$$

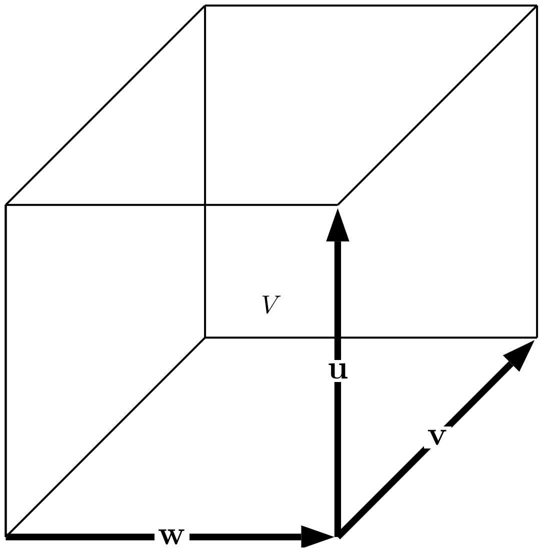
and we have seen that the vector $\mathbf{v} \times \mathbf{w}$ *is orthogonal to* $\mathbf{v}$ *and* $\mathbf{w}$.



Volume in $\mathbb{R}^3$: A parallelepiped

As in the two-dimensional case we get an easy formula for the volume of a parallelepiped spanned by three vectors $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{u}$.

$$V = |\mathbf{v}.(\mathbf{w} \times \mathbf{u})| = |\mathbf{w}.(\mathbf{v} \times \mathbf{u})| = |\mathbf{u}.(\mathbf{v} \times \mathbf{w})|$$



Example 4.8 (Volume Worked Example in $\mathbb{R}^2$).

The area of the parallelogram spanned by the two vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ *is given by*

$$A = \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right| = \|1 \cdot 0 - 1 \cdot 2\| = 2$$

Example 4.9 (Volume Worked Example in $\mathbb{R}^3$).

The volume, V, of the parallelepiped spanned by the three vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$ *is given by:*

$$\begin{aligned} V &= \left| \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} . \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right) \right| \\ &= \left| \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right| \\ &= |-4| \\ &= 4 \end{aligned}$$

Theorem 4.19 (Identities for the vector and the scalar product)**.**

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *and* $\mathbf{x} \in \mathbb{R}^3$. *Then we have the following identities.*

- $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.
- $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\||\sin(\theta)|$.
- $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{u}.\mathbf{v})\mathbf{w}$ *(Grassmann-expansion).*
- $(\mathbf{u} \times \mathbf{v}).(\mathbf{w} \times \mathbf{x}) = (\mathbf{u}.\mathbf{w})(\mathbf{v}.\mathbf{x}) - (\mathbf{v}.\mathbf{w})(\mathbf{u}.\mathbf{x})$. *(Lagrange identity).*

Linear maps, Vectors and Matrices

Definition of linear map

Linear mappings $f : \mathbb{R} \mapsto \mathbb{R}$ are of the form

$$f(x) = ax, \quad a \in \mathbb{R}$$

For example $f(x) = 2x$.

Theorem 4.20 (Properties of a Linear Map).

A linear mapping $f : \mathbb{R} \mapsto \mathbb{R}$ *has the properties*

$$\begin{aligned} f(x+y) &= f(x) + f(y) \quad \textit{for all} \quad x, y \in \mathbb{R} \\ f(cx) &= cf(x) \quad \textit{for all} \quad x \in \mathbb{R}, \quad c \in \mathbb{R} \end{aligned}$$

We will now generalise this idea for n-dimensional vectors.

Definition 4.21 (Linear Map).

Let m *and* n *be positive integers.*

A mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ *with the following properties*

$$\begin{aligned} f(\mathbf{v}+\mathbf{w}) &= f(\mathbf{v}) + f(\mathbf{w}) \quad \textit{for all} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \\ f(k\mathbf{v}) &= kf(\mathbf{v}) \quad \textit{for all} \quad \mathbf{v} \in \mathbb{R}^n, \quad k \in \mathbb{R} \end{aligned}$$

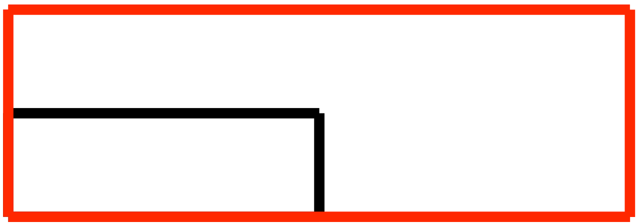
is called a linear map.

Linear Maps in Practice

Computer Graphics and Computer Vision is full of linear maps:

Example 4.10 (Scaling).

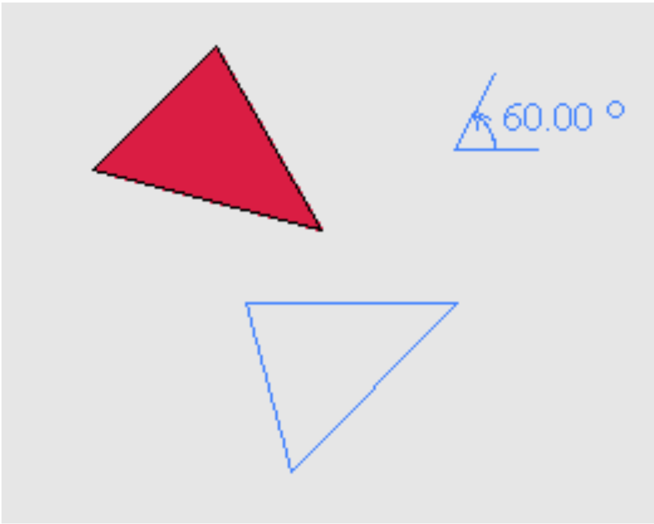
The example $f(\mathbf{v}) = k\mathbf{v}$ *is a scaling:*

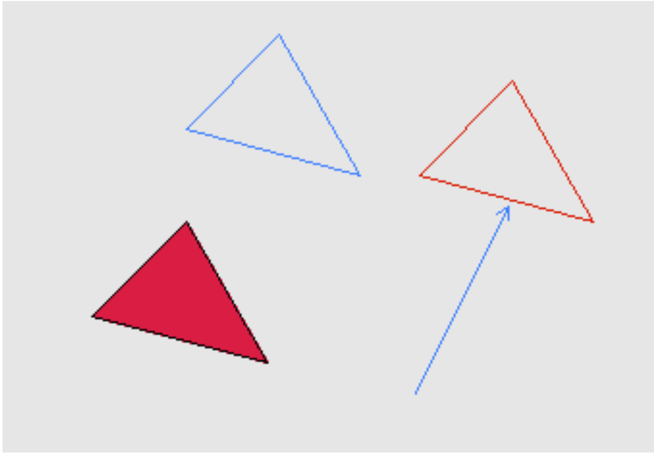


Scaling

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Example 4.11 (Rotation and Translation).





$$\begin{bmatrix} X_{rotated} \\ Y_{rotated} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{translated} \\ Y_{translated} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & D_x \\ 0 & 1 & D_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Example 4.12 (A $\mathbb{R}^2$ Linear Mapping).

The mapping $e : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $e(x, y) = \begin{pmatrix} 3x + 4y \\ x \end{pmatrix}$ *is linear.*

This can be seen as follows:
We take two arbitrary vectors $\mathbf{a} = (a_1, a_2)$ *and* $\mathbf{b} = (b_1, b_2)$ *in* $\mathbb{R}^2$ *and see that*

$$\begin{aligned} e(a_1 + b1, a_2 + b_2) &= \begin{pmatrix} 3(a_1 + b1) + 4(a_2 + b2) \\ a_1 + b_1 \end{pmatrix} \\ &= \begin{pmatrix} 3a_1 + 4a_2 \\ a_1 \end{pmatrix} + \begin{pmatrix} 3b_1 + 4b_2 \\ b_1 \end{pmatrix} \\ &= e(a_1, a_2) + e(b1, b2) \end{aligned}$$

Furthermore if we take an arbitrary real number k *we get*

$$e(ka_1, ka_2) = \begin{pmatrix} 3ka_1 + 4ka_2 \\ ka_1 \end{pmatrix} = k \begin{pmatrix} 3a_1 + 4a_2 \\ a_1 \end{pmatrix}$$

Example 4.13 (A $\mathbb{R}^3 \mapsto \mathbb{R}^2$ Linear Mapping).

The mapping $f : \mathbb{R}^3 \mapsto \mathbb{R}^2$, $f(x, y, z) = \left(\, x + y + z, 0 \,\right)$ *is linear as well.*

Example 4.14 (A **non-linear** mapping).

The mapping $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$*,* $g(x, y) = \begin{pmatrix} x + y \\ y^2 \end{pmatrix}$ *is* **not linear** *:*

- *one easily sees this since a quadratic term appears in the second component.*

To show this formally we take the vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ *and evaluate*

$$g(0, 1) + g(0, 2) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \neq \begin{pmatrix} 3 \\ 9 \end{pmatrix} = g(0 + 0, 1 + 2)$$

Linear Mappings and Vector Scalar Products

We can define a linear mapping $h : \mathbb{R}^n \mapsto \mathbb{R}$ via the scalar product. Let $\mathbf{y}$ be a fixed vector in $\mathbb{R}^n$.

$$h_y(x) = \mathbf{x}.\mathbf{y}$$

is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$.

Theorem 4.22 (Linear Mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^n$)**.**

A mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ *given by*

$$f(v_1, v_2, \ldots, v_n) = \begin{pmatrix} f_1(v_1, v_2, \ldots, v_n) \\ f_2(v_1, v_2, \ldots, v_n) \\ \vdots \\ f_n(v_1, v_2, \ldots, v_n) \end{pmatrix}$$

is **linear if and only if** *all the mappings* $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ *are linear.*

We will soon simplify the treatment of linear mappings. For this we need the following definition.

Definition 4.23 (Vector Standard Basis).
We define the standard basis of $\mathbb{R}^n$ as the set of vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Importance of a Basis: Linear Combination

We can express every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ as a *linear combination* of the standard basis, i.e.

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + v_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Example 4.15 (Simple Vector Basis).

Consider the vector $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \\ 7 \end{pmatrix}$.

We can write $\mathbf{a}$ *as*

$$a = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Generalising this Linear Mapping

In general this reads as follows. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear mapping and let $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$.

Then we have

$$f(\mathbf{v}) = f(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n) = v_1 f(\mathbf{e}_1) + v_2 f(\mathbf{e}_2) + \cdots + v_n f(\mathbf{e}_n)$$

This holds for every vector $\mathbf{v} \in \mathbb{R}^n$.

Thus we can easily calculate $f(\mathbf{v})$ for any $\mathbf{v} \in E^n$ if we know the values of $f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)$.

Definition 4.24 (Matrices).
Let n be a positive integer and $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ a linear mapping given by

$$f(\mathbf{v}) = \begin{pmatrix} f_1(v) \\ f_2(v) \\ \vdots \\ f_m(v) \end{pmatrix}.$$

Then f can be represented by a matrix A where

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

where the entries $a_{i,j}$ of the matrix $\mathbf{A}$ are given by

$$a_{i,j} = f_i(e_j)$$

Some Remarks on Matrices

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

where the entries $a_{i,j}$ of the matrix $\mathbf{A}$ are given by

$$a_{i,j} = f_i(e_j)$$

Note that:

- The first column of $\mathbf{A}$ is just $f(\mathbf{e}_1)$, the second is $f(\mathbf{e}_2)$ and so on.
- The space of matrices representing linear mappings $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is denoted as $\mathbb{R}^{m\times n}$.
- m is the number of rows in the matrix and n is the number of columns.

Example 4.16 (Matrix Mapping).

We consider the mapping $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ *given by*

$$f(v_1, v_2, v_3) = \begin{pmatrix} 4v_1 + 3v_3 \\ v_2 - v_1 \\ v_1 + v_2 + v_3 \end{pmatrix}$$

We have

$$f(1,0,0) = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \quad f(0,1,0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad f(0,0,1) = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Thus the matrix $\mathbf{A}$ *that represents* f *is given by*

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Example 4.17 (Some Common Matrix Mappings).

2D Scaling: $\begin{pmatrix} x_k & 0 \\ 0 & y_k \end{pmatrix}$

2D Rotation:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

2D Shear (x axis): $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

2D Shear (y axis): $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

3D Scaling: $\begin{pmatrix} x_k & 0 & 0 \\ 0 & y_k & 0 \\ 0 & 0 & z_k \end{pmatrix}$

3D Rotation about z axis:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2D Translation (Homogeneous Coords):

$$\begin{pmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix-Vector multiplication

It is usually **much easier** to work with the just matrix $\mathbf{A}$ rather than the linear mapping f. Thus we need the following definition.

Definition 4.25 (Matrix-Vector multiplication).

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector in $\mathbb{R}^n$ we define the matrix-vector product of $\mathbf{A}$ with $\mathbf{x}$ as

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} \sum_{j=1}^{n} a_{1,j} x_j \\ \sum_{j=1}^{n} a_{2,j} x_j \\ \cdots \\ \sum_{j=1}^{n} a_{m,j} x_j \end{pmatrix}.$$

If $\mathbf{A}$ is the representation of the linear mapping f it holds

$$\mathbf{A}\mathbf{x} = f(\mathbf{x})$$

Thus we can evaluate $f(\mathbf{x})$ by evaluating the matrix-vector product $\mathbf{A}\mathbf{x}$.

Example 4.18. *Matrix-Vector multiplication*

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & 5 \\ 0 & -3 & 1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$$

Then

$$\mathbf{Ax} = \begin{pmatrix} 5 \\ -32 \\ -15 \end{pmatrix}$$

Note: The i-th entry in the matrix vector product Ax is given by the scalar product of the i-th row of A and x.

Lemma 4.26 (Adding Two Matrices)**.**

We have the following rules for matrices.

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices in* $\mathbb{R}^{m \times n}$.

We add **two matrices** $\mathbf{A}$ *and* $\mathbf{B}$ *by adding every component of* A *with the corresponding component of* $\mathbf{B}$.

Therefore, if we set $\mathbf{C} = \mathbf{A} + \mathbf{B}$*, then*

$$c_{i,j} = a_{i,j} + b_{i,j}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n$$

Lemma 4.27 (Scalar Multiplication of a Matrix).

The **multiplication of a matrix**, $\mathbf{A}, \in \mathbb{R}^{m \times n}$ *by a* **real number (scalar)**, k, *is similar:*

if we set $\mathbf{C} = k\mathbf{A}$, *then we get*

$$c_{i,j} = ka_{i,j}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n$$

i.e. we **multiply every entry of A with** k.

Example 4.19 (Adding two matrices and scalar multiplication).

$$Let\ A = and\ B =$$

$$Let\ k =$$

Consecutive Linear Mappings

We also want to consider the consecutive application of linear mappings.

Given two linear mappings $f : \mathbb{R}^r \mapsto \mathbb{R}^m$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^r$ we want to calculate

$$(f \circ g)(\mathbf{v}) = f(g(\mathbf{v}))$$

It is easily shown that $f \circ g$ is a linear mapping as well.

This operation can be simplified a great deal with the notation of matrices.

To do this we must learn how to multiply matrices together.

Definition 4.28 (Matrix Multiplication).

Let $f : \mathbb{R}^r \mapsto \mathbb{R}^m$ *and* $g : \mathbb{R}^n \mapsto \mathbb{R}^r$ *be two linear mappings with the matrix representations* $\mathbf{A}$ *for* f *and* $\mathbf{B}$ *for* g.

Then the matrix representation $\mathbf{C}$ *of the linear mapping* $h := f \circ g : \mathbb{R}^n \mapsto E^m$ *is given by*

$$c_{i,j} = \sum_{k}^{n} a_{i,k} b_{k,j} \quad 1 \leq i \leq m, \quad 1 \leq j \leq n$$

We write $\mathbf{C} = \mathbf{AB}$.

Please note: We can only multiply two matrices $\mathbf{A}$ and $\mathbf{B}$ as $\mathbf{AB}$ if the number of columns of $\mathbf{A}$ **equals** the number of rows of $\mathbf{B}$.

Lemma 4.29 (Non-commutative Matrix multiplication).

In general we do not have $\mathbf{AB} = \mathbf{BA}$ *for matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$.

Matrix multiplication is said to be **Non-commutative**.

Example 4.20 (Rotation and Translation).

For a given matrix rotation, R_θ *and a given matrix translation,* T.

The compound *transformation mapping* $R_\theta T \neq T R_\theta$

Example 4.21 (Non-commutative Matrix multiplication).

Let

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 4 & 2 & -1 & 0 \\ -5 & -3 & 1 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 5 & -4 \\ 2 & 2 & 0 & -2 \\ -1 & -3 & 3 & 6 \\ 0 & -4 & -1 & -1 \end{pmatrix}$$

Then

$$\mathbf{AB} = \begin{pmatrix} -3 & -5 & 7 & -8 \\ 5 & 11 & -2 & -9 \\ 9 & 3 & 17 & -26 \\ -12 & -8 & -23 & 31 \end{pmatrix}$$

What is **BA**?

Determinants

The determinant mapping assigns every matrix a real number.

Determinants have many applications:

- Working out Vector Cross products easily.
- Inverting Matrices
- Solving linear systems and
- For other theoretical results concerning linear systems.

Definition 4.30 (Determinants in $\mathbb{R}^2$ and $\mathbb{R}^3$).

Let $\mathbf{A} = (a_{i,j})$ *be a matrix representing a linear mapping* $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$.

We define the **determinant of A** *as*

$$\det \mathbf{A} = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

If $\mathbf{A} = (a_{i,j})$ *is a matrix representing a linear mapping* $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ *we define the determinant of* $\mathbf{A}$ *as*

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \\ &= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} \\ &\quad - a_{1,2}a_{2,1}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} \end{aligned}$$

Example 4.22 (An easy way to calculate determinants).

Definition 4.31 (Determinants in $\mathbb{R}^n$).

Finally we introduce the concept of determinants for general $n \times n$ *matrices.*

We already know that the determinant of a 2×2 *matrix* $\mathbf{A}$ *is given by*

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

and if $\mathbf{A}$ *is a* 3×3 *matrix it is given by*

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Generalising the determinant calculation

Now we want to generalise this concept.

Rewriting the formula above we get

$$\det \mathbf{A} = (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})$$

or

$$\det \mathbf{A} = a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

We see that we can write the determinant of a 3×3 matrix in terms of determinants of 2×2 matrices.

This concept can be generalised to define determinants for general $n \times n$ matrices.

Definition 4.32 (The determinant of an $n \times n$ matrix).

Let $n \geq 2$ *and* $\mathbf{A} = (a_{ij}, 1 \leq i, j \leq n)$ *an* $n \times n$ *matrix.*

The determinant *of* $\mathbf{A}$ *is defined as*

$$\det \mathbf{A} = \sum_{k=1}^{n} (-1)^{1+k} \det \mathbf{A}_{1k},$$

where $\mathbf{A}_{1k}$ *is obtained from* $\mathbf{A}$ *by crossing out row* 1 *and column* k *from* $\mathbf{A}$.

Example 4.23 (Determinant of a 4x4 Matrix).

The determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{pmatrix}$$

is given by

$$\det \mathbf{A} = 1 \det \begin{pmatrix} 0 & 4 & -1 \\ 1 & 0 & 7 \\ 4 & -2 & 0 \end{pmatrix} + 2 \det \begin{pmatrix} 2 & 4 & -1 \\ 3 & 0 & 7 \\ 0 & -2 & 0 \end{pmatrix} + 5 \det \begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 7 \\ 0 & -2 & 0 \end{pmatrix}$$

and so on ...

Matrix Cofactor

So far we have developed the determinant of a matrix $\mathbf{A}$ after the first row, i.e. we used the submatrices $\mathbf{A}_{1k}$ in the calculation of the determinant. However this is not necessarily the best way to do this.

Definition 4.33 (Matrix Cofactor).

Let $\mathbf{A}$ be an $n \times n$ matrix.

The (i, j)-cofactor of A is the **number $\mathbf{C}_{ij}$** *defined as :*

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

where again $\mathbf{A}_{ij}$ is obtained from $\mathbf{A}$ by crossing out the i-th row and the j-th column of $\mathbf{A}$.

Theorem 4.34 (Determinant via cofactor expansion).

The determinant of the matrix $\mathbf{A}$ *can be calculated using the so-called* cofactor expansion *across the* i*-th row :*

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

or across the j*-th column:*

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 4.24 (Determinant via cofactor expansion).

Work out

$$\det \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

Cofactor Expansion of a Triangular Matrix

If $\mathbf{A}$ is a triangular matrix, for example an upper triangular matrix,

- That is a matrix such that $a_{ij} = 0$ if $i > j$

then the cofactor expansion becomes very simple if we use the cofactor expansion across the n-th row, since all but the last entry in that row are 0.

Example 4.25 (Cofactor Expansion of a Triangular Matrix).
Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -4 & 10 \\ 0 & 2 & 3 & -8 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

then

$$\det \mathbf{A} = a_{44} \cdot C_{44}$$

and

$$C_{44} = (-1)^8 \det \begin{pmatrix} 1 & 1 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{pmatrix} = -5 \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = -5 \cdot 2 \cdot 1$$

and thus

$$\det \mathbf{A} = -1 \cdot (-5) \cdot 2 \cdot 1 = 10$$

We see that the determinant is just the product of the entries along the diagonal of **A**.

Determinant of a Triangular Matrix

The same procedure can be applied to lower diagonal matrices by using a cofactor expansion across the first row.

Theorem 4.35 (Determinant of a Triangular Matrix)**.**

If $\mathbf{A}$ *is a triangular matrix, then the determinant of* $\mathbf{A}$ *is the product of the diagonal entries.*

Definition 4.36 (The Adjoint of a Square Matrix).

The adjoint *of a* **square matrix**, $\mathbf{A}$ *is defined as the transpose of a matrix of cofactors of* $\mathbf{A}$.

The adjoint is written as **adj A**

Definition 4.37 (Transpose of a Matrix).

The transpose *of a matrix* $\mathbf{A} = (a_{ij}, 1 \leq i \leq n, 1 \leq j \leq m)$ *(an* $n \times m$ *matrix) is denoted by* $\mathbf{A}^T$ *and is defined as:*

$$\mathbf{A}^{\mathbf{T}} = (a_{ji}, 1 \leq i \leq n, 1 \leq j \leq m) \text{ (an } m \times n \text{ matrix.)}$$

That is you transpose respective rows and columns of the matrix

Example 4.26 (The Adjoint of a Square Matrix).

Work out the adjoint of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$

To find the adjoint of $\mathbf{A}$ we need to:

(a) Form a matrix, $\mathbf{C}$, of all the cofactors of $\mathbf{A}$ as follows:

General Case in $\mathbb{R}^3$:

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

This example:

$$\mathbf{C} = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$$

(b) Write down adj $\mathbf{A} = \mathbf{C}^{\mathbf{T}}$:

$$\mathbf{C}^{\mathbf{T}} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$\mathbf{C}^{\mathbf{T}} = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Vector Cross Product via Determinant/Cofactor

Given two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ we can compute a vector, $\mathbf{v} = (v_1, v_2, v_3)$ as a determinant via:

$$\mathbf{v} = \begin{vmatrix} v1 & v2 & v3 \\ a1 & a2 & a3 \\ b1 & b2 & b3 \end{vmatrix}$$

which is the same as using cofactors::

$$\mathbf{v} = C_{11}v_1 + C_{12}v_2 + C_{13}v_3$$

where $C_1 1$ etc are cofactors of the above determinant.
which is equivalent to saying:

$$\mathbf{v} = (C_{11}, C_{12}, C_{13})$$

Example 4.27. *Vector Cross Product via Determinant/Cofactor Work out the Cross product of the two vectors* $(1, 0, 0)$ *and* $(0, 1, 0)$

Definition 4.38 (The Inverse of a Square Matrix)**.**

The inverse *of a square matrix,* $\mathbf{A}$*, is denoted as* $\mathbf{A}^{-1}$*, and is defined as:*

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

where $\mathbf{I}$ *is the identity matrix.*

E.g. $\mathbf{I}$ *in* $\mathbb{R}^3$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Theorem 4.39 (Calculating the Inverse of a Square Matrix)**.**

Knowing the adjoint of a matrix it is easy to form the Inverse of a Square Matrix, $\mathbf{A}$*:*

(a) Calculate adj $\mathbf{A} = \mathbf{C}^{\mathbf{T}}$ *(in* $\mathbb{R}^3$*):*

$$adj\ \mathbf{A} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

(b) Calculate the determinant of $\mathbf{A}$, $\det \mathbf{A}$ *and form the inverse of* $\mathbf{A}$ *by dividing adj* $\mathbf{A}$ *by* $\det \mathbf{A}$ *(in* $\mathbb{R}^3$*):*

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{C_{11}}{\det \mathbf{A}} & \frac{C_{21}}{\det \mathbf{A}} & \frac{C_{31}}{\det \mathbf{A}} \\ \frac{C_{12}}{\det \mathbf{A}} & \frac{C_{22}}{\det \mathbf{A}} & \frac{C_{32}}{\det \mathbf{A}} \\ \frac{C_{13}}{\det \mathbf{A}} & \frac{C_{23}}{\det \mathbf{A}} & \frac{C_{33}}{\det \mathbf{A}} \end{pmatrix}$$

Example 4.28 (The Inverse of a Square Matrix).

Work out the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$

We already have: adj $\mathbf{A} = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$

The determinant of $\mathbf{A}$ is:

$$\det \mathbf{A} = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix} = 2(0 - 24 - 3(0 - 6) + 5(16 - 1) = 45$$

So the inverse of $\mathbf{A}$ is:

$$\mathbf{A}^{-1} = \tfrac{1}{45} \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Solving a System of Linear Equations

We have seen that we can represent a system of equations as system of a matrix and vectors:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

can be written as:

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A}$ is the matrix $(a_{ij}, 1 \leq i, j \leq n)$ and $\mathbf{x}$ is the vector (x_i) and $\mathbf{b}$ is the vector (b_i).

Solving for $\mathbf{x}$

Now we need to solve for $\mathbf{x}$ so if we multiply both sides of the equation $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^{-1}$ we get:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

so we can solve the system of equations by calculating the inverse A^{-1} and multiplying the vector $\mathbf{b}$ by this.

Example 4.29 (Solving a System of Linear Equations). *Solve to following system of equations:*

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 45 \\ 4x_1 + x_2 + 6x_3 &= 90 \\ x_1 + 4x_2 \; (+0x_3) &= 45 \end{aligned}$$

We can write this as:

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 45 \\ 90 \\ 45 \end{pmatrix}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

We already know $\mathbf{A}^{-1}$ *from a few slides ago*

So solving for $\mathbf{x}$ *we get*

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

which is:

$$\begin{aligned}
\mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{45}\begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}\begin{pmatrix} 45 \\ 90 \\ 45 \end{pmatrix} \\
&= \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 29 \\ 4 \\ -5 \end{pmatrix}
\end{aligned}$$

THE END