Strong Admissibility and Infinite Argumentation Frameworks

Martin Caminada

Cardiff University
CaminadaM@cardiff.ac.uk

Abstract. Strong admissibility plays an important role in formal argumentation under the grounded semantics, especially when explaining the acceptance of an argument. However, strong admissibility has so far only been defined in the context of finite argumentation frameworks. In the current paper, we examine the case of infinite argumentation frameworks. In particular, we assess what the challenges are when moving from finite to infinite argumentation frameworks and we show that despite these challenges, strong admissibility can be meaningfully defined and applied in the context of *finitary* argumentation frameworks.

Keywords: Abstract Argumentation · Strong Admissibility · Infinite Argumentation Frameworks.

1 Introduction

Formal argumentation has become one of the key approaches for symbolic reasoning under uncertainty [1]. Within formal argumentation, strong admissibility [2, 4, 6] plays a key role, especially in the context of grounded semantics. In essence, strong admissibility relates to grounded semantics in a similar way as admissibility relates to preferred semantics, especially when it comes to proof procedures. In order to show that an argument is in a preferred extension, it is not necessary to construct the entire preferred extension. Instead, it is sufficient to show that the argument is in an admissible set. Similarly, in order to show that an argument is in the grounded extension, it is not necessary to construct the entire grounded extension. Instead, it is sufficient to show that the argument is in a strongly admissible set [6]. Such a strongly admissible set can then either be presented in its original form, or be the basis for an interactive explanation in the form of a discussion game [5].

Strong admissibility has so far only been defined for finite argumentation frameworks [2, 4, 6, 9, 8]. This can be a limitation, especially when applying strong admissibility in the context of instantiated argumentation. For instance, when applying ASPIC$^+$ [14] with domain independent strict rules (that is, with strict rules based on classical logic entailment) the mere fact that there exist an infinite number of tautologies implies that there will be an infinite number of arguments. As such, it is worthwhile to explore how the concept of strong admissibility can be applied to infinite argumentation frameworks as well.

In the current paper, we examine the challenges when it comes to applying strong admissibility in the context of infinite argumentation frameworks. We show that for a particular class of infinite argumentation frameworks (called *finitary* argumentation frameworks [13]) it is still possible to apply strong admissibility, in both its set-based form and in its labelling-based form. We show that these forms are equivalent to each other and satisfy the same properties that have previously been proved in the context of finite argumentation frameworks.

The current paper is structured as follows. First, in Section 2, we provide some basic definitions and formal preliminaries. Then, in Section 3 we present some of the existing definitions of strong admissibility and examine why these are problematic in the context of infinite argumentation frameworks. Then, in Section 4 we examine how two of the definitions of strong admissibility (a set-based definition and a labelling-based definition) can still be used in the context of *finitary* argumentation frameworks, and that doing so results in properties similar as in the context of finite argumentation frameworks. We round off in Section 5 with a discussion of the obtained results.

2 Preliminaries

In the current section, we briefly restate some of the key concepts of abstract argumentation theory, in both its extension-based and labelling-based form.

Definition 1. *An* argumentation framework *is a pair* (Ar, att) *where* Ar *is a set of entities, called arguments, whose internal structure can be left unspecified, and* att *is a binary relation on* Ar*. For any* $A, B \in Ar$ *we say that* A attacks B *iff* $(A, B) \in att$*.*

Definition 2. *Let* (Ar, att) *be an argumentation framework,* $A \in Ar$ *and* $\mathcal{A}rgs \subseteq Ar$*. We define* A^+ *as* $\{B \in Ar \mid A \text{ attacks } B\}$*,* A^- *as* $\{B \in Ar \mid B \text{ attacks } A\}$*,* $\mathcal{A}rgs^+$ *as* $\cup\{A^+ \mid A \in \mathcal{A}rgs\}$*, and* $\mathcal{A}rgs^-$ *as* $\cup\{A^- \mid A \in \mathcal{A}rgs\}$*.* $\mathcal{A}rgs$ *is said to be* conflict-free *iff* $\mathcal{A}rgs \cap \mathcal{A}rgs^+ = \emptyset$*.* $\mathcal{A}rgs$ *is said to* defend A *iff* $A^- \subseteq \mathcal{A}rgs^+$*. The characteristic function* $F : 2^{Ar} \rightarrow 2^{Ar}$ *is defined as* $F(\mathcal{A}rgs) = \{A \mid \mathcal{A}rgs \text{ defends } A\}$*.*

Definition 3. *Let* (Ar, att) *be an argumentation framework.* $\mathcal{A}rgs \subseteq Ar$ *is said to be:*

- *an admissible set iff* $\mathcal{A}rgs$ *is conflict-free and* $\mathcal{A}rgs \subseteq F(\mathcal{A}rgs)$
- *a complete extension iff* $\mathcal{A}rgs$ *is conflict-free and* $\mathcal{A}rgs = F(\mathcal{A}rgs)$
- *a grounded extension iff* $\mathcal{A}rgs$ *is the (unique) smallest (w.r.t.* $\subseteq$*) complete extension*
- *a preferred extension iff* $\mathcal{A}rgs$ *is a maximal (w.r.t.* $\subseteq$*) complete extension*

The above definitions essentially follow the extension-based approach of [13].[1] It is also possible to define the key argumentation concepts in terms of argument labellings [3, 7].

[1] In [13] a preferred extension is defined as a maximal admissible set, instead of as a maximal complete extension, but as was first stated in [3], these two characterisations are equivalent.

Definition 4. *Let (Ar, att) be an argumentation framework. An* argument labelling *is a function $\mathcal{L}ab : Ar \to \{\mathtt{in}, \mathtt{out}, \mathtt{undec}\}$. An argument labelling $\mathcal{L}ab$ is called an* admissible labelling *iff for each $A \in Ar$ it holds that:*

- *if $\mathcal{L}ab(A) = \mathtt{in}$ then for each B that attacks A it holds that $\mathcal{L}ab(B) = \mathtt{out}$*
- *if $\mathcal{L}ab(A) = \mathtt{out}$ then there exists a B that attacks A such that $\mathcal{L}ab(B) = \mathtt{in}$*

$\mathcal{L}ab$ is called a complete labelling *iff it is an admissible labelling and for each $A \in Ar$ it also holds that:*

- *if $\mathcal{L}ab(A) = \mathtt{undec}$ then there is a B that attacks A such that $\mathcal{L}ab(B) = \mathtt{undec}$, and for each B that attacks A such that $\mathcal{L}ab(B) \neq \mathtt{undec}$ it holds that $\mathcal{L}ab(B) = \mathtt{out}$*

As a labelling is essentially a function, we sometimes write it as a set of pairs. Also, if $\mathcal{L}ab$ is a labelling, we write $\mathtt{in}(\mathcal{L}ab)$ for $\{A \in Ar \mid \mathcal{L}ab(A) = \mathtt{in}\}$, $\mathtt{out}(\mathcal{L}ab)$ for $\{A \in Ar \mid \mathcal{L}ab(A) = \mathtt{out}\}$ and $\mathtt{undec}(\mathcal{L}ab)$ for $\{A \in Ar \mid \mathcal{L}ab(A) = \mathtt{undec}\}$. As a labelling is also a partition of the arguments into sets of in-labelled arguments, out-labelled arguments and undec-labelled arguments, we sometimes write it as a triplet $(\mathtt{in}(\mathcal{L}ab), \mathtt{out}(\mathcal{L}ab), \mathtt{undec}(\mathcal{L}ab))$.

Definition 5 ([10]). *Let $\mathcal{L}ab$ and $\mathcal{L}ab'$ be argument labellings of an argumentation framework (Ar, att). We say that $\mathcal{L}ab \sqsubseteq \mathcal{L}ab'$ iff $\mathtt{in}(\mathcal{L}ab) \subseteq \mathtt{in}(\mathcal{L}ab')$ and $\mathtt{out}(\mathcal{L}ab) \subseteq \mathtt{out}(\mathcal{L}ab')$. $\mathcal{L}ab \sqcap \mathcal{L}ab'$ is defined as $(\mathtt{in}(\mathcal{L}ab) \cap \mathtt{in}(\mathcal{L}ab'), \mathtt{out}(\mathcal{L}ab) \cap \mathtt{out}(\mathcal{L}ab'), Ar \setminus ((\mathtt{in}(\mathcal{L}ab) \cap \mathtt{in}(\mathcal{L}ab')) \cup (\mathtt{out}(\mathcal{L}ab) \cap \mathtt{out}(\mathcal{L}ab'))))$. $\mathcal{L}ab \sqcup \mathcal{L}ab'$ is defined as $((\mathtt{in}(\mathcal{L}ab) \setminus \mathtt{out}(\mathcal{L}ab')) \cup (\mathtt{in}(\mathcal{L}ab') \setminus \mathtt{out}(\mathcal{L}ab)), (\mathtt{out}(\mathcal{L}ab) \setminus \mathtt{in}(\mathcal{L}ab')) \cup (\mathtt{out}(\mathcal{L}ab') \setminus \mathtt{in}(\mathcal{L}ab)), (\mathtt{in}(\mathcal{L}ab) \cap \mathtt{out}(\mathcal{L}ab') \cup (\mathtt{out}(\mathcal{L}ab) \cap \mathtt{in}(\mathcal{L}ab')) \cup (\mathtt{undec}(\mathcal{L}ab) \cap \mathtt{undec}(\mathcal{L}ab'))))$.*

Definition 6. *Let $\mathcal{L}ab$ be a complete labelling of an argumentation framework (Ar, att). $\mathcal{L}ab$ is said to be*

- *a grounded labelling iff $\mathcal{L}ab$ is the (unique) smallest (w.r.t. $\sqsubseteq$) complete labelling*
- *a preferred labelling iff $\mathcal{L}ab$ is a maximal (w.r.t. $\sqsubseteq$) complete labelling*

Given an argumentation framework (Ar, att) we define two functions Args2Lab and Lab2Args (to translate a conflict-free set of arguments to an argument labelling, and to translate an argument labelling to a set of arguments, respectively) such that $\mathtt{Args2Lab}(\mathcal{A}rgs) = (\mathcal{A}rgs, \mathcal{A}rgs^{+}, Ar \setminus (\mathcal{A}rgs \cup \mathcal{A}rgs^{+}))$ and $\mathtt{Lab2Args}(\mathcal{L}ab) = \mathtt{in}(\mathcal{L}ab)$. It has been proven [7] that if $\mathcal{A}rgs$ is an admissible set (resp. a complete, grounded or preferred extension) then $\mathtt{Args2Lab}(\mathcal{A}rgs)$ is an admissible labelling (resp. a complete, grounded or preferred labelling), and that if $\mathcal{L}ab$ is an admissible labelling (resp. a complete, grounded or preferred labelling) then $\mathtt{Lab2Args}(\mathcal{L}ab)$ is an admissible set (resp. a complete, grounded or preferred extension). Moreover, when the domain and range of Args2Lab and Lab2Args are restricted to complete extensions and complete labellings they become injective functions that are each other's reverses, which implies that the complete extensions (resp. the grounded extension and the preferred extensions) and the complete labellings (resp. the grounded labelling and the preferred labellings) are one-to-one related [7].

3 Strong Admissibility and Infinite Argumentation Frameworks

In the current section, we provide a brief overview of strong admissibility in its various forms, as well as of the challenges one encounters when trying to apply this concept in the context of infinite argumentation frameworks. Due to space limitations, we are unable to provide a general discussion of how strong admissibility is applied for finite argumentation frameworks. For this, we refer the reader to [6].

The concept of strong admissibility was first introduced by Baroni and Giacomin [2], using the notion of *strong defence*.

Definition 7 ([2]). *Let* (Ar, att) *be an argumentation framework,* $A \in Ar$ *and* $\mathcal{A}rgs \subseteq Ar$*.* A *is* strongly defended *by* $\mathcal{A}rgs$ *iff each attacker* $B \in Ar$ *of* A *is attacked by some* $C \in \mathcal{A}rgs \setminus \{A\}$ *such that* C *is strongly defended by* $\mathcal{A}rgs \setminus \{A\}$*.*

Baroni and Giacomin say that a set $\mathcal{A}rgs$ satisfies the strong admissibility property iff it strongly defends each of its arguments [2]. However, it is also possible to define strong admissibility in an equivalent way without having to refer to strong defence [6].

Definition 8 ([6]). *Let* (Ar, att) *be an argumentation framework.* $\mathcal{A}rgs \subseteq Ar$ *is* strongly admissible *iff every* $A \in \mathcal{A}rgs$ *is defended by some* $\mathcal{A}rgs' \subseteq \mathcal{A}rgs \setminus \{A\}$ *which in its turn is again strongly admissible.*

It is important to note that Definition 7 and Definition 8 have so far only been applied in the context of finite argumentation frameworks (that is, argumentation frameworks in which the number of arguments is finite). Unfortunately, these definitions cannot easily be applied in the context where the argumentation framework is infinite. To see why, consider the infinite argumentation framework $AF_1 = (Ar, att)$ where $Ar = \{A_1, A_2, A_3, \ldots\}$ and $att = \{(A_{i+1}, A_i) \mid i \geq 1\}$. This argumentation framework is shown in Figure 1.

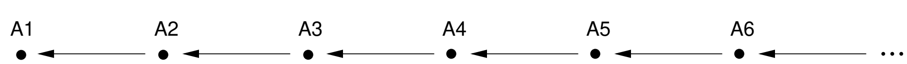

Fig. 1. AF_1: each argument is attacked by its successor

In argumentation framework AF_1 there exist precisely three admissible sets: $\emptyset$, $\{A_i \mid i \text{ is odd }\}$ and $\{A_i \mid i \text{ is even }\}$. The first set is the grounded extension. The second and third set are the preferred extensions. However, when trying to apply either Definition 7 or Definition 8 to assess whether the latter two sets are strongy admissible, one stumbles upon a problem. Take for instance the set $\{A_i \mid i \text{ is odd }\}$. When applying Definition 7 to assess whether A_1 is strongly defended by $\{A_i \mid i \text{ is odd }\}$, we observe that A_1's attacker A_2 is attacked by

$A_3 \in \{A_i \mid i \text{ is odd }\} \setminus \{A_1\}$. So we need to assess whether A_3 is strongly defended by $\{A_i \mid i \text{ is odd }\} \setminus \{A_1\}$. For this, we need to assess whether A_5 is strongly defended by $\{A_i \mid i \text{ is odd }\} \setminus \{A_1, A_3\}$, etc. The point here is that Definition 7 has a recursive nature, and for the argumentation framework AF_1 the recursion does not end. As such, one could either assume that for each odd j, A_j is strongly defended by $\{A_i \mid i \text{ is odd }\} \setminus \{A_k \mid k \text{ is odd and } k < j\}$, or that for each odd j, A_j is *not* strongly defended by $\{A_i \mid i \text{ is odd }\} \setminus \{A_k \mid k \text{ is odd and } k < j\}$. Both assumptions are consistent with Definition 7, yet only one of them can hold.

A similar problem occurs in the context of Definition 8. Here, in order to determine whether $\{A_i \mid i \text{ is odd }\}$ is a strongly admissible set, we have to determine whether A_1 is defended by some subset of $\{A_i \mid i \text{ is odd }\} \setminus \{A_1\}$ which in its turn is strongly admissible. In essence, Definition 8 is another example of a recursive definition of which the recursion does not end for argumentation framework AF_1.

A third definition of strong admissibility was provided in [6, Lemma 2, Theorem 1].[2]

Definition 9. *Let (Ar, att) be an argumentation framework and let $\mathcal{A}rgs \subseteq Ar$. Let $H^0_{\mathcal{A}rgs} = \emptyset$ and $H^{i+1}_{\mathcal{A}rgs} = F(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$ $(i \geq 0)$. $\mathcal{A}rgs$ is strongly admissible iff $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs} = \mathcal{A}rgs$.*

Definition 9 is not recursive. As such, it avoids the problem of potential infinite recursion. In particular, for AF_1 it can be observed that for any set $\mathcal{A}rgs$, $H^0_{\mathcal{A}rgs} = \emptyset$, $H^1_{\mathcal{A}rgs} = F(H^0_{\mathcal{A}rgs}) \cap \mathcal{A}rgs = \emptyset$, $H^2_{\mathcal{A}rgs} = F(H^1_{\mathcal{A}rgs}) \cap \mathcal{A}rgs = \emptyset$, etc. As such, the only set that is strongly admissible is the empty set, which as we observed before, is also the grounded extension.

Although Definition 9 allows one to unambiguously assess, even for infininite argumentation frameworks, whether a particular set is strongly admissible or not, it still has some issues. Consider the argumentation framework $AF_1 = (Ar, att)$ with $Ar = \{A_i \mid i \geq 1\} \cup \{B\}$ and $att = \{(A_i, A_{i+1}) \mid i \geq 1\} \cup \{(A_j, B) \mid j \text{ is even }\}$. This argumentation framework is shown in Figure 2.

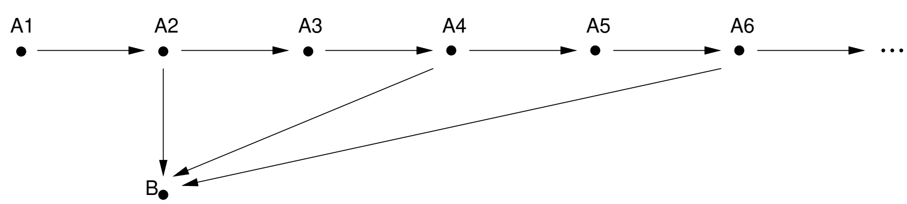

Fig. 2. AF_2: an argumentation framework that is not finitary in the sense of [13]

[2] It has been shown that Definition 7, Definition 8 and Definition 9 are equivalent to each other in the context of finite argumentation frameworks [6].

AF_2 only has one complete extension: $\{A_j \mid j \text{ is odd }\} \cup \{B\}$, which is also the grounded extension. Yet, this grounded extension is not strongly admissible, at least not according to Definition 9. This is because (when taking $\mathcal{A}rgs$ as $\{A_j \mid j \text{ is odd }\} \cup \{B\}$) $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ is $\{A_j \mid j \text{ is odd }\}$ instead of $\{A_j \mid j \text{ is odd }\}$ $\cup \{B\}$.[3] More seriously, even though B is in the grounded extension, there is no strongly admissible set that contains B, at least not according to Definition 9. This is a problem, as the whole idea of strong admissibility is to show that an argument is in the grounded extension by showing that it is in a strongly admissible set [9].[4] For finite argumentation frameworks, this property actually holds; in particular, it also holds that the grounded extension is always strongly admissible. For infinite argumentation frameworks, the property unfortunately does not always hold, as shown by the counter example of AF_2.

Strong admissibility, apart from its set-based form, has also been defined in a labelling-based form. This is done using the concept of a min-max numbering.

Definition 10 ([6]). *Let* $\mathcal{L}ab$ *be an admissible labelling of an argumentation framework* (Ar, att)*. A* min-max numbering *is a total function* $\mathcal{MM}_{\mathcal{L}ab} : \texttt{in}(\mathcal{L}ab) \cup \texttt{out}(\mathcal{L}ab) \rightarrow \mathbb{N} \cup \{\infty\}$ *such that for each* $A \in \texttt{in}(\mathcal{L}ab) \cup \texttt{out}(\mathcal{L}ab)$ *it holds that:*

- *if* $\mathcal{L}ab(A) = \texttt{in}$ *then* $\mathcal{MM}_{\mathcal{L}ab}(A) = max(\{\mathcal{MM}_{\mathcal{L}ab}(B) \mid B \text{ attacks } A \text{ and } \mathcal{L}ab(B) = \texttt{out}\}) + 1$ *(with* $max(\emptyset)$ *defined as* 0*)*
- *if* $\mathcal{L}ab(A) = \texttt{out}$ *then* $\mathcal{MM}_{\mathcal{L}ab}(A) = min(\{\mathcal{MM}_{\mathcal{L}ab}(B) \mid B \text{ attacks } A \text{ and } \mathcal{L}ab(B) = \texttt{in}\}) + 1$ *(with* $min(\emptyset)$ *defined as* ∞*)*

In the context of finite argumentation frameworks, it has been proven that every admissible labelling has a unique min-max numbering [6].

Definition 11. *A* strongly admissible labelling *is an admissible labelling whose min-max numbering yields natural numbers only (so no argument is numbered* ∞*).*

An important limitation is that min-max numberings have only been applied in the context of finite argumentation frameworks. Unfortunately, applying min-max numberings in the context of infinite argumentation frameworks is not straightforward. Consider again the example of AF_2 (Figure 2). Here, there exists only one complete labelling. In this labelling (which is also the grounded labelling) every odd A_i is labelled `in`, every even A_i is labelled `out`, and B is labelled `in`. As for the min-max numbering of this labelling, it can be verified that each A_i will be numbered with i. However, when it comes to numbering B we encounter a problem. The attackers of B are the `out`-labelled arguments A_2, A_4, A_6, etc. These are respectively numbered 2, 4, 6, etc. As B itself is labelled `in`, we have to apply point 1 of Definition 10, which specifies that $\mathcal{MM}_{\mathcal{L}ab}(B) = max(\{2, 4, 6, \ldots\}) + 1$. However the maximum element of the set $\{2, 4, 6, \ldots\}$ is not defined. Therefore, the min-max number of B is not defined, at least not according to Definition 10.

[3] A similar problem was observed in [13] w.r.t. the inductive proof procedure for grounded semantics.

[4] In a similar way, one shows that an argument is in a preferred extension by showing that it is in an admissible set.

4 Strong Admissibility and Finitary Argumentation Frameworks

In the current section, we show how the concept of strong admissibility can be applied in the context of infinite argumentation frameworks. However, we do have to restrict ourselves to argumentation frameworks that are *finitary* [13], meaning that although there can be an infinite number of arguments and an infinite number of attacks, each argument has to have a finite number of attackers.

Definition 12 ([13]). *An argumentation framework* $AF = (Ar, att)$ *is called* finitary *iff for each* $A \in Ar$*, the set* $\{B \mid (B, A) \in att\}$ *is finite.*

It turns out that for finitary argumentation frameworks, Definition 9 yields the same desirable properties as have previously been proved for finite argumentation frameworks [6].

Theorem 1. *Let* (Ar, att) *be a finitary argumentation framework and let* $\mathcal{A}rgs \subseteq Ar$ *be a strongly admissible set (in the sense of Definition 9). It holds that* $\mathcal{A}rgs$ *is an admissible set.*

Proof. We first observe that if $\mathcal{A}rgs'$ is an admissible set then (1) $F(\mathcal{A}rgs')$ is admissible, and (2) $\mathcal{A}rgs' \subseteq F(\mathcal{A}rgs')$. We proceed to show, by induction on i, that $H^i_{\mathcal{A}rgs}$ is admissible.

BASIS Let $i = 0$. In that case, $H^i_{\mathcal{A}rgs} = H^0_{\mathcal{A}rgs} = \emptyset$, which is admissible.

STEP Suppose that $H^i_{\mathcal{A}rgs}$ is admissible. From observation (1) it follows that $F(H^i_{\mathcal{A}rgs})$ is also admissible. We now have to prove that also $F(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$ is admissible. We first observe that $F(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$ is conflict-free, as $F(H^i_{\mathcal{A}rgs})$ is conflict-free by virtue of being admissible. Next, suppose towards a contradiction that $F(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$ does not defend all of its arguments. This means that $F(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$ contains an argument (say A) that has an attacker (say B) that is not attacked by any argument $C \in F(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$. This is in spite of the fact that $F(H^i_{\mathcal{A}rgs})$ does contain at least one attacker of B (follows from observation (1)). It follows that all such attackers are not in $\mathcal{A}rgs$. But then all these attackers are also not in $H^i_{\mathcal{A}rgs}$, which means that $A \notin F(H^i_{\mathcal{A}rgs})$. Contradiction.

From the thus obtained fact that each $H^i_{\mathcal{A}rgs}$ is admissible, observation (2) allows us to infer that $H^i_{\mathcal{A}rgs} \subseteq H^{i+1}_{\mathcal{A}rgs}$ (for each $i \geq 0$). This implies that $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ is conflict-free (as any two attacking $A, B \in \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ would also have to be in some $H^i_{\mathcal{A}rgs}$ $(i \geq 0)$, which conflicts with $H^i_{\mathcal{A}rgs}$ being admissible and conflict-free). It also implies that $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ defends all of its arguments. This can be seen as follows. Let $A \in \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$. Then there exists a $H^i_{\mathcal{A}rgs}$ $(i \geq 0)$ such that $A \in H^i_{\mathcal{A}rgs}$. The fact that $H^i_{\mathcal{A}rgs}$ is admissible means that for each attacker B of A, $H^i_{\mathcal{A}rgs}$ contains a C that attacks B. But then $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ contains the same C. As such, $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ defends all its arguments. This, together with the earlier observed fact that $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ is conflict-free, means that $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ is admissible. As $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs} = \mathcal{A}rgs$, it therefore follows that $\mathcal{A}rgs$ is admissible.

Baroni and Giacomin prove that in the context of finite argumentation frameworks, the grounded extension is the unique biggest (w.r.t. $\subseteq$) strongly admissible set [2]. We proceed to prove that this result still holds in the context of finitary argumentation frameworks.

Theorem 2. *Let $AF = (Ar, att)$ be a finitary argumentation framework. The grounded extension of AF is the biggest (w.r.t. $\subseteq$) strongly admissible set (in the sense of Definition 9) of AF.*

Proof. We first show that the grounded extension is a strongly admissible set. Let GE be the grounded extension. From [13] it follows that $GE = \cup_{i=0}^{\infty} F^i$, where $F^0 = \emptyset$ and $F^{i+1} = F(F^i)$. It directly follows that for each $i \geq 0$, $F^i \subseteq GE$, so $F^i \cap GE = F^i$. This implies that for each $i \geq 0$, $H^i_{GE} = F^i$, so $\cup_{i=0}^{\infty} H^i_{GE} = \cup_{i=0}^{\infty} F^i = GE$, which means that GE is a strongly admissible set.
We proceed to show that GE is also the *biggest* strongly admissible set. Let $\mathcal{A}rgs$ be an arbitrary strongly admissible set. From the fact that $\mathcal{A}rgs$ is strongly admissible, it follows that $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs} = \mathcal{A}rgs$. Suppose $\mathcal{A}rgs \supseteq GE$. Then from $F^i \cap GE = F^i$ it follows that $F^i \cap \mathcal{A}rgs = F^i$. This implies that $H^i_{\mathcal{A}rgs} = F^i$, so $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs} = \cup_{i=0}^{\infty} F^i = GE$. From the fact that $\mathcal{A}rgs$ is strongly admissible, it then follows that $\mathcal{A}rgs = GE$.

In addition to the grounded extension being the biggest strongly admissible set, it can be shown that the empty set is the smallest strongly admissible set.

Proposition 1. *Let $AF = (Ar, att)$ be a finitary argumentation framework. The empty set ($\emptyset$) is the smallest strongly admissible set (in the sense of Definition 9) of AF.*

Proof. This follows from the fact that the empty set is always strongly admissible in the sense of Definition 9, together with the fact that the empty set is a subset of each strongly admissible set.

It can be proved that the strongly admissible sets form a lattice[5] with the grounded extension as its top element (Theorem 2) and the empty set as its bottom element (Proposition 1). This has previously been proved in the context of finite argumentation frameworks [6], but we show that this result still holds in the context of finitary argumentation frameworks.

Proposition 2. *Let $\mathcal{A}rgs$ and $\mathcal{A}rgs'$ be sets of arguments such that $\mathcal{A}rgs \subseteq \mathcal{A}rgs'$. For every $i \geq 0$ it holds that $H^i_{\mathcal{A}rgs} \subseteq H^i_{\mathcal{A}rgs'}$.*

Proof. By induction on i.

[5] We recall that a *lattice* is a partial order such that each two elements have both a greatest lower bound and a least upper bound.

BASIS Let $i = 0$. Then $H^0_{\mathcal{A}rgs} = \emptyset = H^0_{\mathcal{A}rgs'}$.

STEP Suppose that for some i it holds that $H^i_{\mathcal{A}rgs} \subseteq H^i_{\mathcal{A}rgs'}$. As F is a monotonic function, it follows that $F(H^i_{\mathcal{A}rgs}) \subseteq F(H^i_{\mathcal{A}rgs'})$. From the fact that $\mathcal{A}rgs \subseteq \mathcal{A}rgs'$ it then follows that $F(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs \subseteq F(H^i_{\mathcal{A}rgs'}) \cap \mathcal{A}rgs'$. That is, $H^i_{\mathcal{A}rgs} \subseteq H^i_{\mathcal{A}rgs'}$.

Lemma 1. *Let $AF = (Ar, att)$ be a finitary argumentation framework and let $\mathcal{A}rgs_1 \subseteq Ar$ and $\mathcal{A}rgs_2 \subseteq Ar$. If $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ are strongly admissible sets (in the sense of Definition 9), then $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is also a strongly admissible set (in the sense of Definition 9).*

Proof. Suppose $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ are strongly admissible. That is, $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs_1} = \mathcal{A}rgs_1$ and $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs_2} = \mathcal{A}rgs_2$. We now proceed to prove that $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2} = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$.

"$\subseteq$" By definition, it holds for each $i \geq 0$ that $H^i_{\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2} \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$, which implies that $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2} \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$.

"$\supseteq$" Let $A \in \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. Then either $A \in \mathcal{A}rgs_1$ or $A \in \mathcal{A}rgs_2$. Assume without loss of generality that $A \in \mathcal{A}rgs_1$ (the case of $A \in \mathcal{A}rgs_2$ is similar). From the fact that $\mathcal{A}rgs_1 = \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs_1}$ it follows that $A \in \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs_1}$. This means there exists an $i \geq 0$ such that $A \in H^i_{\mathcal{A}rgs_1}$. As $\mathcal{A}rgs_1 \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs2$, we can apply Proposition 2 to obtain that $H^i_{\mathcal{A}rgs_1} \subseteq H^i_{\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2}$, so $A \in H^i_{\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2}$. This directly implies that $A \in \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2}$.

Lemma 2. *Let $AF = (Ar, att)$ be a finitary argumentation framework. Each set of arguments $\mathcal{A}rgs \subseteq Ar$ has a unique biggest (w.r.t. $\subseteq$) strongly admissible (in the sense of Definition 9) subset.*

Proof. We first observe that there is always at least one strongly admissible set (the empty set). We also observe that every increasing sequence of strongly admissible sets $\mathcal{A}rgs_1, \mathcal{A}rgs_2, \mathcal{A}rgs_3, \ldots$ has an upper bound ($\cup_{i=1}^{\infty} \mathcal{A}rgs_i$ which is again strongly admissible; this follows from Lemma 1). This allows us to apply Zorn's lemma and obtain that there is at least one maximal strongly admissible set.[6] We now proceed to show that this maximal strongly admissible subset is unique. Let $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ be maximal strongly admissible subsets of $\mathcal{A}rgs$. Now consider $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. From Lemma 1 it follows that this is again a strongly admissible set. From the fact that $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ are *maximal* strongly admissible subsets, it follows that if $\mathcal{A}rgs_1 \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ then $\mathcal{A}rgs_1 = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$, and that if $\mathcal{A}rgs_2 \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ then $\mathcal{A}rgs_2 = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$, so we obtain that $\mathcal{A}rgs_1 = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ and $\mathcal{A}rgs_2 = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ so $\mathcal{A}rgs_1 = \mathcal{A}rgs_2$.

Theorem 3. *Let AF be a finitary argumentation framework. The strongly admissible sets (in the sense of Definition 9) of AF form a lattice (w.r.t. $\subseteq$).*

[6] Although not explicitly mentioned in [13], a similar form of reasoning is needed to prove that maximal admissible sets (i.e. preferred extensions) always exist, even for an infinite argumentation framework with an infinite sequences of ever increasing admissible sets.

Proof. This can be proved in a similar way as Theorem 5 of [6], although the lemmas used in this proof would need to be replaced by Lemma 1 and Lemma 2, as the latter apply in the context of finitary argumentation frameworks instead of finite argumentation frameworks.

As for the labelling-based definition of strong admissibility, we observe that when restricting ourselves to finitary argumentation frameworks, the concept of a min-max numbering is always well-defined. This is because, in Definition 10, the maximal element of a set of numbers is always defined as long as this set is finite. Although the existing proofs in [6] were developed in the context of a finite argumentation framework, they do not actually rely on this, as long as the concept of a min-max numbering is well-defined. This means the existing proofs in [6] carry over to finitary argumentation frameworks in a straightforward way.

Theorem 4. *Let $AF = (Ar, att)$ be a finitary argumentation framework and let $\mathcal{L}ab$ be an admissible labelling of AF. $\mathcal{L}ab$ has a unique min-max numbering.*

Proof. Similar to the proof of Theorem 6 of [6].

Theorem 5. *Let $AF = (Ar, att)$ be a finitary argumentation framework.*

- *for every strongly admissible set $\mathcal{A}rgs$ of AF (in the sense of Definition 9), it holds that* $\texttt{Args2Lab}(\mathcal{A}rgs)$ *is a strongly admissible labelling*
- *for every strongly admissible labelling $\mathcal{L}ab$ of AF, it holds that* $\texttt{Lab2Args}(\mathcal{L}ab)$ *is a strongly admissible set (in the sense of Definition 9)*

Proof. Similar to the proof of Theorem 7 of [6].

We proceed to show that the grounded labelling is the biggest strongly admissible labelling and that the all-`undec` labelling[7] is the smallest strongly admissible labelling.

Theorem 6. *Let $AF = (Ar, att)$ be a finitary argumentation framework. The grounded labelling of AF is the biggest (w.r.t. $\sqsubseteq$) strongly admissible labelling of AF.*

Proof. Let $\mathcal{A}rgs$ be the grounded extension of AF and let $\mathcal{L}ab$ be $\texttt{Args2Lab}(\mathcal{A}rgs)$. From [7, Definition 9 and Theorem 6] it follows that $\mathcal{L}ab$ is the grounded labelling. From Theorem 5 and the fact that the grounded extension is strongly admissible (Theorem 2), it follows that $\mathcal{L}ab$ is a strongly admissible labelling. The next thing to show is that $\mathcal{L}ab$ is also the *biggest* (w.r.t. $\sqsubseteq$) strongly admissible labelling. Let $\mathcal{L}ab'$ be a strongly admissible labelling. Then Theorem 5 implies that $\mathcal{A}rgs' = \texttt{Lab2Args}(\mathcal{L}ab')$ is a strongly admissible set. As the grounded extension is the biggest strongly admissible set (Theorem 2), it holds that $\mathcal{A}rgs' \subseteq \mathcal{A}rgs$, so $\texttt{in}(\mathcal{L}ab') \subseteq \texttt{in}(\mathcal{L}ab)$. From [7, Lemma 1] it follows that $\texttt{out}(\mathcal{L}ab') \subseteq \texttt{out}(\mathcal{L}ab)$, so it follows that $\mathcal{L}ab' \sqsubseteq \mathcal{L}ab$. This, together with our initial assumption that $\mathcal{L}ab \sqsubseteq \mathcal{L}ab'$ implies that $\mathcal{L}ab' = \mathcal{L}ab$.

[7] The all-`undec` labelling labels each argument `undec`.

Proposition 3. *Let $AF = (Ar, att)$ be a finitary argumentation framework. The all-*`undec` *labelling of AF is the smallest (w.r.t. $\sqsubseteq$) strongly admissible labelling of AF.*

Proof. From Definition 4 it follows that the all-`undec` labelling is admissible. Its min-max numbering is empty, as there are no `in` or `out` labelled arguments to be numbered. This trivially implies that no argument is numbered ∞. Hence, the all-`undec` labelling is strongly admissible. It is also the *smallest* strongly admissible labelling, as for each strongly admissible labelling $\mathcal{L}ab'$ it holds that $\mathcal{L}ab \sqsubseteq \mathcal{L}ab'$, with $\mathcal{L}ab$ being the all-`undec` labelling.

We proceed to show that the strongly admissible labellings form a lattice with the grounded labelling as its top element (Theorem 6) and the all-`undec` labelling as its bottom element (Proposition 3). Notice that the mere fact that the strongly admissible sets form a lattice does by itself not directly imply that the strongly admissible labellings also form a lattice, as the relationship between strongly admissible sets and strongly admissible labellings is one-to-many instead of one-to-one.[8] Still, the proofs are very similar.

Lemma 3. *Let $AF = (Ar, att)$ be a finitary argumementation framework. If $\mathcal{L}ab_1$ and $\mathcal{L}ab_2$ are strongly admissible labellings, then $\mathcal{L}ab_1 \sqcup \mathcal{L}ab_2$ is also a strongly admissible labelling.*

Proof. Similar to the proof of Lemma 5 of [6]

Lemma 4. *Let $AF = (Ar, att)$ be a finitary argumentation framework. Each admissible labelling $\mathcal{L}ab$ of AF has a unique biggest (w.r.t. $\sqsubseteq$) strongly admissible sublabelling.*

Proof. Similar to the proof of Lemma 2, but with labellings instead of sets and $\subseteq$ replaced by $\sqsubseteq$ and $\cup$ replaced by $\sqcup$.

Theorem 7. *Let AF be a finitary argumentation framework. The strongly admissible labellings of AF form a lattice (w.r.t. $\sqsubseteq$).*

Proof. This can be proved similar to Theorem 5 of [6], with $\subseteq$ replaced by $\sqsubseteq$, $\cup$ replaced by $\sqcup$, $\cap$ replaced by $\sqcap$, and by using the labelling-specific results of Lemma 3 and Lemma 4 instead of their set-specific variants.

5 Discussion

In essence, the current work generalises the results in [6], regarding both the well-definedness and the properties of strong admissibility, in both its set-based form and its labelling-based form. In particular, we have shown that for finitary argumentation frameworks, the concept of strong admissibility is well-defined

[8] We refer to [6] for an example.

(using Definition 9, as well as Definition 10 and Definition 11) and satisfies the same properties that were previously shown for finite argumentation frameworks.

As for the practical applicability of our results, we could look at the field of instantiated argumentation formalisms. For instance, in Assumption-Based Argumentation (ABA) [11] each argument is written as $Asms \vdash c$, where $Asms$ is a set of assumptions that allows one to infer conclusion c. This inference in essence takes the form of a tree of ABA rules (similar to how inferences work in for instance ASPIC^{+}) [14]. If one would take the set of ABA rules to coincide with all possible classical logic entailments (as was for instance done in [12]), one would obtain an infinite set of rules and an infinite set or arguments, as there would for instance be an argument $\emptyset \vdash t$ for each tautology t. However, as long as the set of assumptions is finite,[9] each argument will have a finite number of assumptions and a finite number of attackers. As such, the resulting argumentation framework is *finitary*, which means that we can apply the concept of strong admissibility as discussed in the current paper. That is, in order to show that an argument is in the grounded extension, we do not have to show the entire grounded extension (which would be infinite). Instead, it suffices to show that the argument is in a strongly admissible set.[10]

In terms of how the theory in the current paper relates to what was previously been developed regarding strong admissibility, we can make the following observations.

1. We have loosened the restriction on the argumentation frameworks under which the concept is defined (from *finite* argumentation frameworks to *finitary* argumentation frameworks).
2. Our theory is backwards compatible, meaning that for finite argumentation frameworks, a set of arguments is strongly admissible (Definition 9) iff it is strongly admissible according to the definitions that only work for finite argumentation frameworks (Definition 7 and Definition 8).
3. The strongly admissible sets (and labellings) form a lattice with the empty set (all-`undec` labelling) at the bottom and the grounded extension (grounded labelling) at the top.

One could imagine a further broadening of the concept of strong admissibility, which, instead of from finite to finitary, would go from finitary to unrestricted. Ideally, such a broadening would satisfy similar properties as those mentioned above. That is, such a theory would relate to finitary argumentation frameworks in a similar way as our theory relates to finite argumentation frameworks (point 2). How to construct such a theory is a topic for further research.

Acknowledgments. This publication is supported in part by the Joint Research and Innovation Seed Grants Program between Cardiff University and the University of Illinois System.

[9] Additionally, we would also need to require that for each assumption the set of its contraries is finite.

[10] In essence, showing that an argument is in a strongly admissible set can be done by the kind of tree-based proof procedures that are also applied in ABA [6].

References

1. Baroni, P., Gabbay, D., Giacomin, M., van der Torre, L.: Handbook of Formal Argumentation, vol. 1. College Publications (2018)
2. Baroni, P., Giacomin, M.: On principle-based evaluation of extension-based argumentation semantics. Artificial Intelligence **171**(10-15), 675–700 (2007)
3. Caminada, M.: On the issue of reinstatement in argumentation. In: Fischer, M., van der Hoek, W., Konev, B., Lisitsa, A. (eds.) Logics in Artificial Intelligence; 10th European Conference, JELIA 2006. pp. 111–123. Springer (2006), lNAI 4160
4. Caminada, M.: Strong admissibility revisited. In: Parsons, S., Oren, N., Reed, C., Cerutti, F. (eds.) Computational Models of Argument; Proceedings of COMMA 2014. pp. 197–208. IOS Press (2014)
5. Caminada, M.: Argumentation semantics as formal discussion. In: Handbook of Formal Argumentation, vol. 1, pp. 487–518. College Publications (2018)
6. Caminada, M., Dunne, P.: Strong admissibility revisited: theory and applications. Argument & Computation **10**, 277–300 (2019)
7. Caminada, M., Gabbay, D.: A logical account of formal argumentation. Studia Logica **93**(2-3), 109–145 (2009), special issue: new ideas in argumentation theory
8. Caminada, M., Harikrishnan, S.: An evaluation of algorithms for strong admissibility. In: Proceedings of SAFA 2024. pp. 69–82 (2024)
9. Caminada, M., Harikrishnan, S.: Tractable algorithms for strong admissibility. Argument & Computation (2024), in print
10. Caminada, M., Pigozzi, G.: On judgment aggregation in abstract argumentation. Autonomous Agents and Multi-Agent Systems **22**(1), 64–102 (2011)
11. Čyras, K., Fan, X., Schulz, C., Toni, F.: Assumption-based argumentation: Disputes, explanations, preferences. In: Handbook of Formal Argumentation, vol. 1. College Publications (2018)
12. Dimopoulos, Y., Nebel, B., Toni, F.: On the Computational Complexity of Assumption-based Argumentation for Default Reasoning. Artificial Intelligence **141**(1–2), 57–78 (Oct 2002)
13. Dung, P.: On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. Artificial Intelligence **77**, 321–357 (1995)
14. Modgil, S., Prakken, H.: The ASPIC+ framework for structured argumentation: a tutorial. Argument & Computation **5**, 31–62 (2014), special Issue: Tutorials on Structured Argumentation