Semi-Stable Semantics [1]

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Abstract. In this paper, we examine an argument-based semantics called *semi-stable semantics*. Semi-stable semantics is quite close to traditional stable semantics in the sense that every stable extension is also a semi-stable extension. One of the advantages of semi-stable semantics is that there exists at least one semi-stable extension. Furthermore, if there also exists at least one stable extension, then the semi-stable extensions coincide with the stable extensions. This, and other properties, make semi-stable semantics an attractive alternative for the more traditional stable semantics, which until now has been widely used in fields such as logic programming and answer set programming.

Keywords. argumentation frameworks, argument based semantics, stable semantics, preferred semantics

1. Introduction

In the field of argumentation and defeasible reasoning, stable semantics is one of the oldest ways of determining which arguments or statements can be considered as justified. Well-known examples of formalisms in which stable semantics is applied are default logic [9] and stable models of logic programs [6]. Although alternative semantics have been stated over the years, like for instance grounded semantics which has its origins in Pollock's OSCAR [8] and in the well-founded semantics of logic programming [10], stable semantics has kept considerable support and is currently used even in relatively modern fields such as Answer Set Programming [7].

The popularity of stable semantics is not entirely without reason. It is a quite simple and straightforward semantics in which every argument is assigned a status of either `in` or `out` [2]. Furthermore, it is also a very credulous semantics in the sense that the intersection of the stable extensions is a superset of the intersection of the preferred extensions, which is in its turn a superset of the grounded extension. In some domains, like using argumentation for belief revision, one may prefer to use a credulous approach.

Nevertheless, stable semantics has its shortcomings, of which the potential absence of stable extensions is the most obvious one. Preferred semantics has been proposed as an alternative [5], but it has as a side effect that additional non-stable extensions can be introduced, even in situations where stable extensions already exist. An interesting question is whether one could find a semantics that is "backward compatible" to stable semantics in the sense that it is equivalent to stable semantics in situations where stable extensions exist and still yields a reasonable result (preferably quite close to stable) in

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situations where stable extensions do not exist. In this paper we show that a relatively simple and straightforward principle can be used to form the basis of such a semantics. We propose this semantics as a practical alternative for domains and applications where stable semantics is still being applied.

2. Basic Definitions

We first start with some basic definitions regarding abstract argumentation based on [5].

Definition 1 (argumentation framework). *An argumentation framework is a pair* (Ar, def) *where* Ar *is a finite set of arguments and* $def \subseteq Ar \times Ar$.

The shorthand notation A^+ and A^- stands for, respectively, the set of arguments defeated by A and the set of arguments that defeat A. If $\mathcal{A} \subseteq Ar$ then we write $(Ar, def)_{|\mathcal{A}}$ as a shorthand for $(\mathcal{A}, \{(A, B) \mid (A, B) \in def \text{ and } A, B \in \mathcal{A}\})$.

Definition 2 (defense / conflict-free). *Let* $A \in Ar$ *and* $\mathcal{A}rgs \subseteq Ar$.
We define A^+ *as* $\{B \mid A\ def\ B\}$ *and* $\mathcal{A}rgs^+$ *as* $\{B \mid A\ def\ B$ *for some* $A \in \mathcal{A}rgs\}$.
We define A^- *as* $\{B \mid B\ def\ A\}$ *and* $\mathcal{A}rgs^-$ *as* $\{B \mid B\ def\ A$ *for some* $A \in \mathcal{A}rgs\}$.
$\mathcal{A}rgs$ defends *an argument* A *iff* $A^- \subseteq \mathcal{A}rgs^+$.
$\mathcal{A}rgs$ *is* conflict-free *iff* $\mathcal{A}rgs \cap \mathcal{A}rgs^+ = \emptyset$.

In the following definition, $F(\mathcal{A}rgs)$ stands for the set of arguments that are acceptable (in the sense of [5]) with respect to $\mathcal{A}rgs$. Notice that the definitions of grounded, preferred and stable semantics are provided in terms of complete semantics, which has the advantage of making the proofs in the remainder of this paper more straightforward. Although these definitions are different from the ones provided by Dung [5], it is proved in the appendix that they are in fact equivalent to Dung's versions of grounded, preferred and stable semantics.

Definition 3 (acceptability semantics). *Let* $\mathcal{A}rgs$ *be a conflict-free set of arguments and* $F : 2^{\mathcal{A}rgs} \rightarrow 2^{\mathcal{A}rgs}$ *be a function with* $F(\mathcal{A}rgs) = \{A \mid A$ *is defended by* $\mathcal{A}rgs\}$.

- $\mathcal{A}rgs$ *is* admissible *iff* $\mathcal{A}rgs \subseteq F(\mathcal{A}rgs)$.
- $\mathcal{A}rgs$ *is a* complete *extension iff* $\mathcal{A}rgs = F(\mathcal{A}rgs)$.
- $\mathcal{A}rgs$ *is a* grounded *extension iff* $\mathcal{A}rgs$ *is the minimal (w.r.t. set-inclusion) complete extension.*
- $\mathcal{A}rgs$ *is a* preferred *extension iff* $\mathcal{A}rgs$ *is a maximal (w.r.t. set-inclusion) complete extension.*
- $\mathcal{A}rgs$ *is a* stable *extension iff* $\mathcal{A}rgs$ *is a complete extension that defeats every argument in* $Ar \backslash \mathcal{A}rgs$.

Note that there is only one grounded extension. It contains all the arguments which are not defeated, as well as those arguments which are directly or indirectly defended by non-defeated arguments.

We say that an argument is *credulously justified* under a particular semantics iff it is in at least one extension under this semantics. We say that an argument is *sceptically justified* under a particular semantics iff it is in each extension under this semantics.

3. Semi-Stable Semantics

The notion of semi-stable semantics, as put forward in the current paper, is quite similar to that of preferred semantics. The only difference is that not $\mathcal{A}rgs$ is maximized, but $\mathcal{A}rgs \cup \mathcal{A}rgs^+$.

Definition 4. *Let (Ar, def) be an argumentation framework and $\mathcal{A}rgs \subseteq Ar$. $\mathcal{A}rgs$ is called a* semi-stable extension *iff $\mathcal{A}rgs$ is a complete extension where $\mathcal{A}rgs \cup \mathcal{A}rgs^+$ is maximal.*

If $\mathcal{A}rgs$ is a complete extension, then $\mathcal{A}rgs \cup \mathcal{A}rgs^+$ is called its *range* — a notion first introduced by Bart Verheij [11].

The first thing to notice is that every stable extension is also a semi-stable extension.

Theorem 1. *Let $\mathcal{A}rgs$ be a stable extension of argumentation framework (Ar, def). $\mathcal{A}rgs$ is also a semi-stable extension of (Ar, def).*

Proof. Let $\mathcal{A}rgs$ be a stable extension of (Ar, def). Then $\mathcal{A}rgs$ is a complete extension that defeats every argument in $Ar \backslash \mathcal{A}rgs$. This means that $\mathcal{A}rgs \cup \mathcal{A}rgs^+ = Ar$. Therefore, $\mathcal{A}rgs \cup \mathcal{A}rgs^+$ is maximal (it cannot be a proper superset of Ar). Therefore, $\mathcal{A}rgs$ is a semi-stable extension. □

The converse of Theorem 1 does not hold. That is, it is not the case that each semi-stable extension is also a stable extension. This is illustrated by the following example.

Example 1. *Let (Ar, def) be an argumentation framework with $Ar = \{A, B, C, D\}$ and $def = \{(A, A), (A, C), (B, C), (C, D)\}$. A graphical representation is shown in figure 1. Here, $\{B, D\}$ is a semi-stable extension which is not a stable extension.*

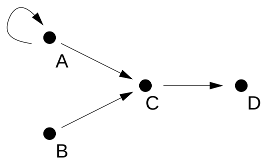

Figure 1. $\{B, D\}$ is a semi-stable but not a stable extension.

Another interesting property of semi-stable semantics is that every semi-stable extension is also a preferred extension.

Theorem 2. *Let $\mathcal{A}rgs$ be a semi-stable extension of argumentation framework (Ar, def). Then $\mathcal{A}rgs$ is also a preferred extension of (Ar, def).*

Proof. Let $\mathcal{A}rgs$ be a semi-stable extension of (Ar, def). Suppose $\mathcal{A}rgs$ is not a preferred extension of (Ar, def). Then there exists a set $\mathcal{A}rgs' \supsetneq \mathcal{A}rgs$ such that $\mathcal{A}rgs'$ is a complete extension. But from $\mathcal{A}rgs' \supsetneq \mathcal{A}rgs$ it follows that $\mathcal{A}rgs'^+ \supseteq \mathcal{A}rgs^+$. Therefore, $(\mathcal{A}rgs' \cup \mathcal{A}rgs'^+) \supsetneq (\mathcal{A}rgs \cup \mathcal{A}rgs^+)$. But then $\mathcal{A}rgs$ would not be a semi-stable extension, since $\mathcal{A}rgs \cup \mathcal{A}rgs^+$ would not be maximal. Contradiction. □

The converse of Theorem 2 does not hold. That is, it is not the case that every preferred extension is also a semi-stable extension. This is illustrated by the following example.

Example 2. *Let (Ar, def) be an argumentation framework with $Ar = \{A, B, C, D, E\}$ and $def = \{(A, B), (B, A), (B, C), (C, D), (D, E), (E, C)\}$. A graphical representation is shown in figure 2. Here, $\{A\}$ is a preferred extension which is not a semi-stable extension. The only semi-stable extension is $\{B, D\}$.*

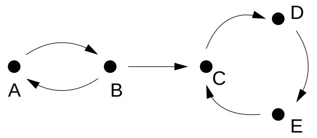

Figure 2. $\{A\}$ is a preferred but not a semi-stable extension.

The overall position of semi-stable semantics is shown in figure 3. Each stable extension is a semi-stable extension; each semi-stable extension is a preferred extension; each preferred extension is a complete extension and the grounded extension is a complete extension.

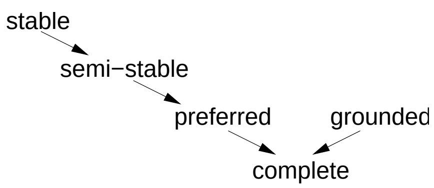

Figure 3. A brief overview of argument based semantics.

It is interesting to observe that in argumentation frameworks where there exists at least one stable extension, the semi-stable extensions coincide with the stable extensions.

Theorem 3. *Let (Ar, def) be an argumentation framework that has at least one stable extension. Let $SE = \{SE_1, \ldots, SE_n\}$ be the set of stable extensions and let $SSE = \{SSE_1, \ldots, SSE_m\}$ be the set of semi-stable extensions. It holds that $SE = SSE$.*

Proof. We need to prove that:

1. $SE \subseteq SSE$
   This follows directly from Theorem 1.
2. $SSE \subseteq SE$
   Let $SE_i \in SE$ (such an SE_i exists since it is assumed that (Ar, def) has at least one stable extension). It holds that $SE_i \cup SE_i^+ = Ar$. Therefore, every semi-stable extension SSE_i will also have to satisfy that $SSE_i \cup SSE_i^+ = Ar$ (otherwise $SSE_i \cup SSE_i^+$ would not be maximal). This means that every semi-stable extension is also a stable extension.

□

For every argumentation framework there exists at least one semi-stable extension. This is because there exists at least one complete extension, and a semi-stable extension is simply a complete extension in which some property (the union of itself and the arguments it defeats) is maximal.

Apart from the guaranteed existence of extensions, semi-stable semantics has yet another advantage to stable semantics. In determining whether an argument is sceptically or credulously justified with respect to semi-stable semantics, one only has to take into account arguments that are *relevant*.

Definition 5. *Let (Ar, def) be an argumentation framework. An argument $A \in Ar$ is* relevant *with respect to an argument $B \in Ar$ iff there exists an undirected path between A and B.*

In stable semantics, irrelevant arguments can influence whether an argument is justified or not. This is illustrated by the following example.

Example 3. *Let (Ar, def) be an argumentation framework with $Ar = \{A, B, C, D\}$ and $def = \{(A, A)\}(B, C), (C, D)\}$. A graphical representation is shown in figure 4. Here, arguments B, C and D are relevant with respect to each other, and argument A is not relevant with respect to B, C and D. Yet, argument A is the reason why there is no stable extension containing B and D.*

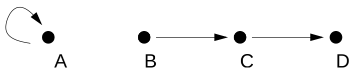

Figure 4. Stable semantics does not satisfy relevance.

Semi-stable semantics, however, does satisfy relevance. Irrelevant arguments have no influence whatsoever on the question whether an argument is justified under semi-stable semantics. To prove this, we first state two lemmas.

Lemma 1. *Let (Ar, def) be an argumentation framework, let $A \in Ar$ and $\mathcal{A} \subseteq Ar$ such that $\mathcal{A}$ is the set of arguments relevant with respect to A. If $\mathcal{A}rgs$ is a semi-stable extension of (Ar, def) then $\mathcal{A}rgs \cap \mathcal{A}$ is a semi-stable extension of $(Ar, def)_{|\mathcal{A}}$.*

Proof. Let $\mathcal{A}rgs$ be a semi-stable extension of (Ar, def). Suppose $\mathcal{A}rgs \cap \mathcal{A}$ is not a semi-stable extension of $(Ar, def)_{|\mathcal{A}}$. Then there exists a complete extension $\mathcal{A}rgs'$ of $(Ar, def)_{|\mathcal{A}}$ with $(\mathcal{A}rgs' \cap \mathcal{A}rgs'^{+}) \supsetneq (\mathcal{A}rgs \cap \mathcal{A}) \cup (\mathcal{A}rgs \cap \mathcal{A})^{+}$. As $\mathcal{A}$ is the largest (w.r.t. set inclusion) set of arguments that are relevant to each other, it holds that $(\mathcal{A}rgs \cap \mathcal{A}) \cup (\mathcal{A}rgs \cap \mathcal{A})^{+} = (\mathcal{A}rgs \cup \mathcal{A}rgs^{+}) \cap \mathcal{A}$. But then $\mathcal{A}rgs$ could not be a semi-stable extension because $\mathcal{A}rgs \cup (\mathcal{A}rgs' \backslash \mathcal{A})$ would be a complete extension with a larger range. Contradiction. □

Lemma 2. *Let (Ar, def) be an argumentation framework, let $A \in Ar$ and $\mathcal{A} \subseteq Ar$ such that $\mathcal{A}$ is the set of arguments relevant with respect to A. If $\mathcal{A}rgs$ is a semi-stable extension of $(Ar, def)_{|\mathcal{A}}$ then there exists a semi-stable extension $\mathcal{A}rgs'$ of (Ar, def) with $\mathcal{A}rgs' \cap \mathcal{A} = \mathcal{A}rgs$.*

Proof. Let $\mathcal{A}rgs$ be a semi-stable extension of $(Ar, def)_{|\mathcal{A}}$. Suppose there exists no semi-stable extension $\mathcal{A}rgs'$ of (Ar, def) with $\mathcal{A}rgs' \cap \mathcal{A} = \mathcal{A}rgs$. Then every complete extension $\mathcal{A}rgs'$ of (Ar, def) with $\mathcal{A}rgs' \cap \mathcal{A} = \mathcal{A}rgs$ does not have a maximal range. Let $\mathcal{A}rgs'$ be a complete extension of (Ar, def), with $\mathcal{A}rgs' \cap \mathcal{A} = \mathcal{A}rgs$, such that $\mathcal{A}rgs' \backslash \mathcal{A}$ is a semi-stable extension of $(Ar, def)_{|(Ar\backslash\mathcal{A})}$. Such an extension always exists since the arguments in $\mathcal{A}$ are not relevant with respect to the arguments in $Ar\backslash\mathcal{A}$. The fact that $\mathcal{A}rgs'$ is not a semi-stable extension of (Ar, def) means that there exists a complete extension with a bigger range. As the range of $\mathcal{A}rgs' \backslash \mathcal{A}$ is already maximal in $(Ar, def)_{|(Ar\backslash\mathcal{A})}$ this can only mean that the range of $\mathcal{A}rgs' \cap \mathcal{A}$ is not maximal in $(Ar, def)_{|\mathcal{A}}$. But as $\mathcal{A}rgs' \cap \mathcal{A} = \mathcal{A}rgs$ this means that $\mathcal{A}rgs$ would not be a semi-stable extension of $(Ar, def)_{|\mathcal{A}}$. Contradiction. □

Theorem 4. *Let (Ar, def) be an argumentation framework and let $A \in Ar$ and $\mathcal{A} \subset Ar$ such that $\mathcal{A}$ is the set of arguments that is relevant with respect to A.*

1. *There exists a semi-stable extension of (Ar, def) iff there exists a semi-stable extension of $(Ar, def)_{|\mathcal{A}}$.*
2. *A is in every semi-stable extension of (Ar, def) iff A is in every semi-stable extension of $(Ar, def)_{|\mathcal{A}}$.*

Proof. This follows directly from Lemma 1 and Lemma 2. □

As each semi-stable extension is also a preferred extension, a straightforward way of computing semi-stable semantics would be to compute all preferred extensions (using an algorithm like [12]) and then to determine which of these are also semi-stable. If one is only interested in whether an argument A is credulously or sceptically justified under semi-stable semantics, one does not have to take into account the entire argumentation framework. Instead, as stated by Theorem Theorem 4, one only has to take into account the arguments that are relevant with respect to A when calculating the preferred extensions. In many cases, however, there also exist alternative ways of determining whether an argument is credulously or sceptically justified under semi-stable semantics.

Theorem 5. *Let (Ar, def) be an argumentation framework, and let $A \in Ar$.*

1. *If A is in the grounded extension, then A is in every semi-stable extension.*
2. *If A is not part of an admissible set, then A is not in any semi-stable extension.*
3. *If A is part of an admissible set but is not defeated by any admissible set then there exists a semi-stable extension containing A.*

Proof.

1. This follows from the fact that the grounded extension is a subset of each complete extension [5], and the fact that each semi-stable extension is a complete extension.
2. This follows from the fact that each semi-stable extension is an admissible set.
3. The fact that A is not defeated by an admissible set also means that A is not defeated by a complete extension, and therefore that A is also not defeated by a semi-stable extension. That is, for any semi-stable extension $\mathcal{A}rgs$, it holds that $A \notin \mathcal{A}rgs^+$. The fact that A is part of an admissible set means that there is a preferred extension containing A. Let $\mathcal{A}rgs'$ be a preferred extension that

contains A and where (within the constraint that it contains A) $\mathcal{A}rgs' \cup \mathcal{A}rgs'^{+}$ is maximal. As for any semi-stable extension $\mathcal{A}rgs$ it holds that $A \notin \mathcal{A}rgs^{+}$, it also holds for any semi-stable extension not containing A that $A \notin \mathcal{A}rgs \cup \mathcal{A}rgs^{+}$. Thus, $\mathcal{A}rgs' \cup \mathcal{A}rgs'^{+}$ cannot be enlarged without losing A. Therefore, $\mathcal{A}rgs'$ is a semi-stable extension.

□

An example of point 3 of Theorem 5 can be found in Figure 2. Here, argument D is in an admissible set but is not defeated by an admissible set. This is because its only defeater (C) is not part of any admissible set. Hence, D is part of a semi-stable extension.

4. Discussion and Research Issues

The idea of semi-stable semantics is not entirely new. It is quite similar to Verheij's concept of an *admissible stage extension*, which fits within Verheij's general approach of using *stages* to deal with the issue of argument reinstatement [11].

Definition 6. *An* admissible stage extension *is a pair* $(\mathcal{A}rgs, \mathcal{A}rgs^{+})$ *where* $\mathcal{A}rgs$ *is an admissible set of arguments and* $\mathcal{A}rgs \cup \mathcal{A}rgs^{+}$ *is maximal.*

It can be shown that Verheij's approach of admissible stage extensions is in fact equivalent to the notion of a semi-stable semantics. This is stated and proved by Proposition 3 in the appendix.

Verheij also studied the relation between stable, semi-stable and preferred semantics, but has done so in terms of his stages approach, which received little following. This, and the fact that his work was published in a relatively small local conference has caused his work not to receive the attention that one may argue it should have received.

Semi-stable semantics can be seen as having a quite natural place within Dung's traditional semantics. One possible way of looking at the issue of argument reinstatement is to label each argument either `in`, `out` or `undec` according to the following postulate.

Postulate 1 ([2])**.** *An argument is labelled* `in` *iff all its defeaters are labelled* `out`*. An argument is labelled* `out` *iff it has a defeater that is labelled* `in`*.*

It can be shown that labellings satisfying this postulates coincide with complete extensions [2]. Furthermore, for labellings that satisfy Postulate 1 it holds that (1) those in which `in` is maximized coincide with preferred extensions, (2) those in which `out` is maximized coincide with preferred extensions, (3) those in which `undec` is maximized coincide with the grounded extension, (4) those in which `in` is minimized coincide with the grounded extension, and (5) those in which `out` is minimized coincide with the grounded extension. Semi-stable extensions then coincide with labellings in which `undec` is minimized (6).

One possible application of semi-stable semantics would be in the field Answer Set Programming [7]. The implementation of semi-stable semantics with respect to Answer Set Programming, however, involves more than just a change at the level of the abstract semantics. As logic programming, of which the Answer Set Programming approach can be seen as a special instance, can be regarded from the perspective of abstract argumentation [5,3], the most obvious way of implementing semi-stable semantics would be at the level of the argumentation framework. Recent research, however, indicates that this may not be enough, since there is an issue regarding the potential violation of argumentation quality postulates [4,1]. For the well-founded semantics, this issue can be dealt with by stating syntactical restrictions on the content of the extended logic program in question [3]. One of our research aims is to study whether a similar approach is also possible in the context of semi-stable semantics.

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Appendix

Proposition 1. *Let (Ar, def) be an argumentation framework and let $\mathcal{A}rgs \subseteq Ar$. The following statements are equivalent:*

1. *$\mathcal{A}rgs$ is the grounded extension*
2. *$\mathcal{A}rgs$ is a minimal fixpoint of F*

Proof.

from 1 to 2: Let $\mathcal{A}rgs$ be the grounded extension. Suppose that $\mathcal{A}rgs$ is not a minimal fixpoint of F. Then there exists a proper subset $\mathcal{A}rgs' \subsetneq \mathcal{A}rgs$ which is a fixpoint of F. As $\mathcal{A}rgs$ is already the smallest fixpoint of F that is conflict-free, this can only mean that $\mathcal{A}rgs'$ is not conflict-free. But this is impossible as a subset of a conflict-free set is also conflict-free. Contradiction.

from 2 to 1: Let $\mathcal{A}rgs$ be a minimal fixpoint of F. As a monotonic increasing function like has a unique minimal fixpoint, the minimal fixpoint of F must be unique. From the previous point of this proof it then follows that the grounded extension is equivalent to this fixpoint.

□

Proposition 2. *Let (Ar, def) be an argumentation framework and let $\mathcal{A}rgs \subseteq Ar$. The following statements are equivalent:*

1. *$\mathcal{A}rgs$ is a preferred extension*
2. *$\mathcal{A}rgs$ is a maximal admissible set*

Proof. This follows from Theorem 25 of [5]. □

Proposition 3. *Let (Ar, def) be an argumentation framework and $\mathcal{A}rgs \subseteq Ar$. The following statements are equivalent:*

1. *$\mathcal{A}rgs$ is a semi-stable extension*
2. *$\mathcal{A}rgs$ is an admissible set of which $(\mathcal{A}rgs, \mathcal{A}rgs^+)$ maximal*

Proof.

from 2 to 1: A complete extension is a stronger condition than an admissible set, so we only need to prove that an admissible set $\mathcal{A}rgs$ where $\mathcal{A}rgs \cup \mathcal{A}rgs^+$ is maximal is also a complete extension. Suppose this is not the case. Then there must be an argument $B \notin \mathcal{A}rgs$ that is defended by $\mathcal{A}rgs$. This means that every argument C that defeats B is defeated by an argument in $\mathcal{A}rgs$. Therefore, $B \notin \mathcal{A}rgs^+$ (otherwise $\mathcal{A}rgs$ would not be conflict-free). This means that $\mathcal{A}rgs \cup \{B\}$ is conflict-free and self-defending, and thus an admissible set. But this would mean that $\mathcal{A}rgs$ is not an admissible set for which $\mathcal{A}rgs \cup \mathcal{A}rgs^+$ is maximal. Contradiction.

from 1 to 2: An admissible set is a weaker condition than a complete extension. We therefore only need to prove that maximality still holds under this weaker condition. Suppose that $\mathcal{A}rgs \cup \mathcal{A}rgs^+$ would not be maximal. This means there exists an admissible set $\mathcal{A}rgs'$ such that $(\mathcal{A}rgs' \cup \mathcal{A}rgs'^+) \supsetneq (\mathcal{A}rgs \cup \mathcal{A}rgs^+)$. From the previous point ("from 2 to 1") it follows that $\mathcal{A}rgs'$ would be a complete extension. But then $\mathcal{A}rgs$ would not have been a complete extension where $\mathcal{A}rgs \cup \mathcal{A}rgs^+$ is maximal. Contradiction.

□

Proposition 4. *Let (Ar, def) be an argumentation framework and let $\mathcal{A}rgs \subseteq Ar$. The following statements are equivalent:*

1. *$\mathcal{A}rgs$ is a stable extension*
2. *$\mathcal{A}rgs$ is a preferred extension that defeats every argument in $Ar \backslash \mathcal{A}rgs$*
3. *$\mathcal{A}rgs$ is an admissible set that defeats every argument in $Ar \backslash \mathcal{A}rgs$*
4. *$\mathcal{A}rgs$ is a conflict-free set that defeats every argument in $Ar \backslash \mathcal{A}rgs$*

Proof.

from 1 to 2: Let $\mathcal{A}rgs$ be a stable extension. This means that $\mathcal{A}rgs$ is a complete extension that defeats every argument in $Ar \backslash \mathcal{A}rgs$. Suppose that $\mathcal{A}rgs$ is not a preferred extension. That means that there is a complete extension $\mathcal{A}rgs' \supsetneq \mathcal{A}rgs$. But as $\mathcal{A}rgs$ defeats every argument in $Ar \backslash \mathcal{A}rgs$, this means that $\mathcal{A}rgs'$ would not be conflict-free and therefore could not be a complete extension. Contradiction.

from 2 to 1: Trivial (every preferred extension is also a complete extension).

from 2 to 3: From Theorem 2 it follows that a preferred extension is a (maximal) admissible set.

from 3 to 2: Let $\mathcal{A}rgs$ be an admissible set that defeats all arguments in $Ar \backslash \mathcal{A}rgs$. Suppose that $\mathcal{A}rgs$ is not a preferred extension. This means that there exists an admissible set $\mathcal{A}rgs' \supsetneq \mathcal{A}rgs$. But as $\mathcal{A}rgs$ defeats all arguments in $Ar \backslash \mathcal{A}rgs$, this would mean that $\mathcal{A}rgs'$ is not conflict-free and therefore could not be an admissible set. Contradiction.

from 3 to 4: This follows directly from the fact that an admissible set is conflict-free.

from 4 to 3: Let $\mathcal{A}rgs$ be a conflict-free set that defeats all arguments in $Ar \backslash \mathcal{A}rgs$. Then, every argument that defeats $\mathcal{A}rgs$ is also defeated by $\mathcal{A}rgs$. This means that $\mathcal{A}rgs$ is an admissible set.

□