CARDIFF UNIVERSITY
EXAMINATION PAPER

Academic Year: 2002/2003
Examination Period: SPRING
Examination Paper Number: CM0188
Examination Paper Title: Analytic Mathematics (II)
Duration: Two hours

Do not turn this page over until instructed to do so by the Senior Invigilator.

Structure of Examination Paper:

There are FOUR pages.
There are FOUR questions in total.
There are no appendices.

The maximum mark for the examination paper is 60 marks = 100% (the equivalent of three complete questions) and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Students to be provided with:

The following items of stationery are to be provided:

ONE answer book.

Instructions to Students:

Answer THREE questions.

The use of translation dictionaries between English or Welsh and a foreign language bearing an appropriate departmental stamp is permitted in this examination.
1. For a positive integer \( n \) let \( \omega \) be a complex *primitive* \( n \)th root of unity, i.e. \( \omega^n = 1 \) and \( \omega^r \neq 1 \) for \( 0 < r \leq n-1 \).

(i) By considering the sum of a GP or otherwise show that for all \( z \in \mathbb{C} \)

\[
1 + z + z^2 + \ldots + z^{n-1} = \begin{cases} 
\frac{z^n - 1}{z - 1} & \text{if } z \neq 1, \\
\frac{n}{z} & \text{if } z = 1.
\end{cases}
\]

(ii) Putting \( z = \omega \) in the above formula show that

\[
1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0
\]

(iii) Suppose \( m \) is not a multiple of \( n \). By putting \( z = \omega^m \) in the formula of part (i) show that \( \omega^m \) (\( 0 \leq m \leq n-1 \)) are all of the \( n \)th roots of unity and that

\[
(\omega^0)^m + (\omega^1)^m + (\omega^2)^m + \ldots + (\omega^{n-1})^m = 0
\]

[Hint: first show \( \omega^m \neq 1 \).]

(iv) Now suppose \( m \) is a multiple of \( n \), say \( m = kn \) where \( k \in \mathbb{N} \). Show that

\[
(\omega^0)^m + (\omega^1)^m + (\omega^2)^m + \ldots + (\omega^{n-1})^m = n
\]

(v) Hence, using (iii) and (iv) with \( n = 3 \), show that the sum of every third term of the series

\[\exp(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}\]

is

\[
\sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!} = \frac{1}{3} \left( \exp(x) + \exp(\omega x) + \exp(\omega^2 x) \right)
\]

where \( \omega \) is a complex cube root of unity.
2. (a) (i) Defining any notation used, state the binomial theorem for the expansion of $(1 + x)^n$ for $n \in \mathbb{N}$. [2]

(ii) Let $u_r$ and $u_{r-1}$ indicate the $r$th term and the $(r-1)$th term respectively in the expansion in increasing powers of $x$ of $(1 + x)^n$. Determine an expression, in its simplest form, for the ratio $u_r/u_{r-1}$. [3]

(iii) Hence show in the binomial expansion of $(1 + 0.01)^8$ the $r$th term is less than $1/100$ the $(r-1)$th term if $r > 4$. [2]

(iv) Hence calculate by hand $(1.01)^8$ correct to three decimal places. [3]

(b) Solve the simultaneous equations

$$5x + 2y \equiv 2 \pmod{11}$$
$$7x + y \equiv 3 \pmod{11}$$

in the field of integers (mod 11). [5]

(c) Find the following limit if it exists

$$\lim_{x \to 0} \frac{(\cos x - 1)}{x^2}$$

[5]

3. (a) For what values of $x$ are the following series convergent

(i) $\sum_{n=1}^{\infty} nx^n$ 
(ii) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$

(iii) $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ 
(iv) $\sum_{n=1}^{\infty} \frac{1}{\log_e(n+1)} x^n$

In each case determine what the appropriate general test shows and for values of $x$ not covered by the general test decide convergence/divergence by other means. [8]

(b) Locate the maxima and minima of the function $y = 2x^3 - 9x^2 + 12x + 6$ and hence make a sketch of the curve. [3]

(c) Differentiate $(\sin 2x)/(x^2 + 2)$ with respect to $x$. [3]

(d) Prove by induction De Moivre’s theorem: For positive integral $n \geq 1$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

[Assume any trigonometric formulae you may require, but these should be clearly stated.]
(a) Briefly explain by means of a diagram how the expression
\[ \int_{x=a}^{b} f(x) \, dx \]
is defined in terms of a limiting process involving upper and lower sums. [7]

(b) First determine where the curves \( y = 2x^2 + 3 \) and \( y = 10x - x^2 \) intersect. Next make an accurate sketch of the two curves. Finally find the area between the curves. [6]

(c) Show (by substitution or otherwise) that
\[ \int \frac{dx}{1 + x^2} = \arctan x + C \]
where \( C \) is an arbitrary constant. Hence by expanding \((1 + x^2)^{-1}\) as a power series in \( x \) and integrating term-by-term derive a power series for \( \arctan x \). Deduce that
\[ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \] [7]
1. For a positive integer $n$ let $\omega$ be a complex primitive $n$th root of unity, i.e. $\omega^n = 1$ and $\omega^k \neq 1$ for $0 < k < n-1$.

(i) By considering the sum of a GP or otherwise show that for all $z \in \mathbb{C}$
\[
1 + z + z^2 + \ldots + z^{n-1} = \begin{cases} 
\frac{z^n - 1}{z - 1} & \text{if } z \neq 1, \\
n & \text{if } z = 1. 
\end{cases}
\]

The sum of the GP
\[
1 + z + z^2 + \ldots + z^{n-1}
\]
with $n$ terms and common ratio $z$ is given by
\[
1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1} \quad \text{if } z \neq 1
\]

If $z = 1$ the sum is obviously $n$ and the required result follows.

(ii) Putting $z = \omega$ in (\*) show that
\[
1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0
\]

Now putting $z = \omega \neq 1$ in (\*) we obtain
\[
1 + \omega + \omega^2 + \ldots + \omega^{n-1} = \frac{\omega^n - 1}{\omega - 1} = 0
\]
as required.

(iii) Suppose $m$ is not a multiple of $n$. By putting $z = \omega^m$ in (\*) show that $\omega^m (0 \leq m \leq n-1)$ are all of the $n$th roots of unity and that
\[
(\omega^0)^m + (\omega^1)^m + (\omega^2)^m + \ldots + (\omega^{n-1})^m = 0
\]
Plainly $(\omega^m)^a = (\omega^a)^m$ for $0 \leq a < m$, so $\omega^m$ is an $n$th root of unity. Since $\omega$ is a primitive $n$th root of unity the numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are all distinct. But by Gauss’ theorem $z^n = 1$ has exactly $n$ roots. Hence the complex numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega$ is any primitive $n$th root of unity, must be all of the roots of the equation.

If $m$ is not a multiple of $n$ then by the remainder theorem it must be of the form $m = hn + r$, where $h$ is an integer and $0 < r < n-1$. But then $\omega^m = \omega^{hn+r} = \omega^r = 1$ since $\omega$ is a primitive $n$th root of unity and $0 < r < n-1$. Hence putting $z = \omega^m = 1$ in (\*) we obtain
\[
1^m + (\omega^1)^m + (\omega^2)^m + \ldots + (\omega^{n-1})^m = \frac{\omega^{nm} - 1}{\omega^r - 1}
\]
\[
= \frac{(\omega^r)^m - 1}{\omega^m - 1} = 0 \quad \text{since } \omega^r = 1, \text{ and } \omega^m \neq 1.
\]

(iv) Now suppose $m$ is a multiple of $n$, say $m = kn$ where $k \in \mathbb{N}$. Show that
\[
(\omega^0)^m + (\omega^1)^m + (\omega^2)^m + \ldots + (\omega^{n-1})^m = n
\]
If $m = kn$ then $\omega^m = \omega^{kn} = (\omega^n)^k = 1$ since $\omega^n = 1$. Thus by (*) with $z = \omega^m = 1$ we have
\[(\omega^m)^0 + (\omega^m)^1 + \ldots + (\omega^m)^{n-1} = (\omega^0)^m + (\omega^1)^m + (\omega^2)^m + \ldots + (\omega^{n-1})^m = n\]
as required.

(v) Hence, using (iii) and (iv) with \(n = 3\), show that the sum of **every third term** of the series

\[\text{Exp}(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}\]
is

\[\sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!} = \frac{1}{3} \left( \text{Exp}(x) + \text{Exp}(\omega x) + \text{Exp}(\omega^2 x) \right)\]

where \(\omega\) is a complex cube root of unity.

For \(n = 3\) either of \(\omega = (-1 + i \sqrt{3})/2\) or \((-1 - i \sqrt{3})/2\) is a primitive cube root of unity. Hence, using the power series expansion for the exponential function and absolute convergence over the whole complex plane, we have

\[\text{Exp}(x) + \text{Exp}(\omega x) + \text{Exp}(\omega^2 x) = \sum_{r=0}^{\infty} \sum_{h=0}^{h=3} \frac{(\omega^h)^r}{h!}\]

\[= \sum_{k=0}^{\infty} \frac{x^{3k}}{h!} = 3 \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!}\]
since \(1^h + \omega^h + \omega^2^h\) is 0, if \(h\) is not a multiple of 3 by (iii), and is 3, if \(h\) is a multiple of 3, say \(h = 3k\), by (iv). The result then follows.

**Not asked.** As a matter of interest we note that

\[\text{Exp}(\omega x) = \exp \left( \frac{-1 + i \sqrt{3}}{2} x \right) = e^{-\frac{x}{2}} \left( \cos \frac{\sqrt{3}}{2} x + i \sin \frac{\sqrt{3}}{2} x \right)\]

\[\text{Exp}(\omega^2 x) = \exp \left( \frac{-1 - i \sqrt{3}}{2} x \right) = e^{-\frac{x}{2}} \left( \cos \frac{\sqrt{3}}{2} x - i \sin \frac{\sqrt{3}}{2} x \right)\]

Hence

\[\frac{1}{3} \left( \text{Exp}(x) + \text{Exp}(\omega x) + \text{Exp}(\omega^2 x) \right) = \frac{1}{3} \text{Exp}(x) + \frac{2}{3} e^{-\frac{x}{2}} \cos \frac{\sqrt{3}}{2} x\]

which is (of course) real if \(x\) is real.
2. (a) (i) Defining any notation used state the binomial theorem for the expansion of \((1 + x)^n\) for \(n \in \mathbb{N}\).

\[
(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + x^n
\]

where

\[
\binom{n}{r} = \frac{n!}{r!(n - r)!} \quad (0 \leq r \leq n)
\]

(ii) Let \(u_r\) and \(u_{r-1}\) indicate the \(r\)th term and the \((r-1)\)th terms respectively in the expansion of \((1 + x)^n\). Determine an expression, in its simplest form, for the ratio \(u_r/u_{r-1}\).

\[
\frac{u_r}{u_{r-1}} = \frac{n!}{r!(n - r)!} \cdot x^r \cdot \frac{(r-1)!(n - r + 1)!}{n!} \cdot \frac{1}{x^{r-1}}
\]

\[
= \frac{(n - r + 1)}{r} x
\]

(iii) Hence show in the binomial expansion of \((1 + 0.01)^8\) the \(r\)th term is less than 1/100 the \((r-1)\)th if \(r > 4\).

If \(n = 8\) taking \(x = 1/100\) we have

\[
\frac{u_r}{u_{r-1}} = \frac{(8 - r + 1)}{r} \cdot \frac{1}{100} \cdot \left(\frac{9}{r} - 1\right) \cdot \frac{1}{100}
\]

\[
\leq 0.8 \times \frac{1}{100} = 0.008 < \frac{1}{100} \quad \text{(for } r \geq 5\text{)}
\]

(iv) Hence calculate by hand \((1.01)^8\) correct to three decimal places.

Hence

\[
(1 + 0.01)^8 = 1 + 8 \times \frac{1}{100} + 28 \times \frac{1}{10000} + 56 \times \frac{1}{1000000} + \ldots + \left(\frac{1}{100}\right)^8
\]

\[
= 1 + 0.08 + 0.0028 + 0.000056 = 1.08286 \quad \text{(rounding to 5dp)}
\]

All terms corresponding to \(r > 4\) are less than 1/100 the preceding term and so cannot affect this approximation.

(b) Solve the simultaneous equations

\[
5x + 2y \equiv 2 \quad (\text{mod } 11)
\]

\[
7x + \quad y \equiv 3 \quad (\text{mod } 11)
\]

in the field of integers (mod 11).

Multiplying the second equation by 2 and remembering to perform arithmetic (mod 11) we obtain
5x + 2y ≡ 2 (mod 11)
3x + 2y ≡ 6 (mod 11)

Subtracting the second from the first we have 2x ≡ 7 (mod 11) which has the solution x = 9 whence y = 6 (mod 11).

(c) Find the following limit if it exists

\[
\lim_{x \to 0} \frac{(\cos x - 1)}{x^2}
\]

Using the power series for \( \cos x \) we have

\[
\frac{(\cos x - 1)}{x^2} = \frac{-x^2}{2!} + \frac{x^4}{4!} + \ldots = -\frac{1}{2} + O(x^2) \quad \text{as } x \to 0
\]

Hence \( \lim (x \to 0) (\cos x - 1)/x^2 = -1/2. \) (Of course many other acceptable ways of finding this limit.)
3. a) For what values of \( x \) are the following series convergent

\[
(i) \quad \sum_{n=1}^{\infty} nx^n \\
(ii) \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n + 1)!} \\
(iii) \quad \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} \\
(iv) \quad \sum_{n=1}^{\infty} \frac{1}{\log(n+1)} x^n
\]

State the test used in each case and determine convergence/divergence in cases where the test fails.

**Ratio test.** Suppose \( a_n \ (n = 1, 2, 3, \ldots) \) is a sequence such that

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \alpha
\]

then

(A) If \( \alpha < 1 \) then the series is convergent.
(B) If \( \alpha > 1 \) then the series is divergent.
(C) If \( \alpha = 1 \) then we can say nothing (the series may or may not converge).

(i) \( |x| < 1 \) - Ratio test and diverges at \( x = 1 \).
(ii) All \( x \in \mathbb{R} \) - Ratio test.
(iii) \( |x| \leq 1 \) - Ratio test and converges at \( x = 1 \) by comparison with \( \sum 1/n^2 \) and \( x = -1 \) by alternating series theorem.
(iv) \( |x| < 1 \) - Ratio test and diverges at \( x = 1 \) by comparison with \( \sum 1/n \). Converges at \( x = -1 \) by the alternating series theorem.

(b) Locate the maxima and minima of the function \( y = 2x^3 - 9x^2 + 12x + 6 \) and roughly sketch the curve.

Differentiating \( y = 2x^3 - 9x^2 + 12x + 6 \) we obtain \( 6x^2 - 18x + 12 \). To obtain the maxima and minima we set this equal to zero. Cancelling throughout by 6 we obtain

\[
x^2 - 3x + 2 = (x - 1)(x - 2) = 0
\]

Thus we have extreme points at \( x = 1, y = 11 \) and \( x = 2, y = 10 \). Using the second derivative test or simply by noting that as \( x \to \infty \) then \( y \to \infty \) and as \( x \to -\infty \) then \( y \to -\infty \) we find a maximum at \( x = 1 \) and a minimum at \( x = 2 \). Figure 1 is a sketch of the graph.

![Figure 1 Sketch of cubic.](image)
(c) Differentiate \((\sin 2x)/(x^2 + 2)\) with respect to \(x\).

We can write this as \((\sin 2x)(x^2 + 2)^{-1}\) so taking \(u = \sin 2x\) and \(v = (x^2 + 2)^{-1}\) and using the product rule we have

\[
\frac{d}{dx}\left((\sin 2x)(x^2 + 2)^{-1}\right)
= 2(\cos 2x)(x^2 + 2)^{-1} + (\sin 2x)(-1)(x^2 + 2)^{-2}\cdot 2x
\]

(d) Prove by induction De Moivre’s theorem: For positive integral \(n \geq 1\)

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\]

[Assume any trigonometric formulae you may require, but these should be clearly stated.]

For \(n = 1\) the result is certainly true. If the result holds for \(n = k\) we have

\[(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta\]

On multiplying both sides by \(\cos \theta + i \sin \theta\) we obtain

\[(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta)(\cos k\theta + i \sin k\theta)\]

\[= (\cos \theta \cos k\theta - \sin \theta \sin k\theta) + i(\sin \theta \cos k\theta + \cos \theta \sin k\theta)\]

If we now use the addition formula for cosine and sine, i.e.

\[
\cos(A + B) = \cos A \cos B - \sin A \sin B
\]
\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]

with \(A = \theta\) and \(B = k\theta\) we obtain

\[(\cos \theta + i \sin \theta)^{k+1} = \cos(k+1)\theta + i \sin(k+1)\theta\]

Hence if the result is true for \(n = k\) then the result is true for \(n = k + 1\). However, we have seen the result is true for \(n = 1\). Hence by the principle of induction the result is true for all \(n \geq 1\).
4. \[ a \]

The area under a curve \( y = f(x) \) which is written as
\[
\int_{a}^{b} f(x) \, dx
\]
is defined by a limiting process. We divide up the interval \([a, b]\) into a finite number of sub-intervals whose maximum length is \( \delta \) (say). Next we form the upper and lower rectangles as shown in Figure 2.

For a partition into \( n \) sub-intervals with \( x_0 = a \) and \( x_n = b \) with the maximum length of the sub-intervals being \( \delta \), we write
\[
L(\delta) = \sum_{i=0}^{n-1} \left( \min_{x \in [x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i)
\]
\[
U(\delta) = \sum_{i=0}^{n-1} \left( \max_{x \in [x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i)
\]

Then
\[
L(\delta) \leq U(\delta)
\]
where \( L(\delta) \) is the sum of the areas of the lower rectangles and \( U(\delta) \) is the sum of the areas of the upper rectangles. (There is no problem in defining the area of a rectangle in the usual way.)

As \( \delta \to 0 \) one can show (under reasonable conditions on \( f \)) that \( L(\delta) \) is monotonic increasing and \( U(\delta) \) is monotonic decreasing. These functions are both bounded so that
\[
\lim_{\delta \to 0} L(\delta) \quad \text{and} \quad \lim_{\delta \to 0} U(\delta)
\]
both exist.

**Definition.** If both limits exists and are equal we define their common value to be
\[
\int_{a}^{b} f(x) \, dx
\]
This is called the integral of \( f \) from \( a \) to \( b \).

[Some reasonable summary of this is sufficient]
(b) First make a sketch of the two curves \( y = 2x^2 + 3 \) and \( y = 10x - x^2 \) showing where they intersect. Next find the area between the curves.

![Graph of two curves with the region of interest shaded.](image)

**Figure 3. Region between the two curves.**

The curves intersect when \( 2x^2 + 3 = 10x - x^2 \) which solving the quadratic gives \( x = 1/3 \) and \( x = 3 \). The area between the two curves is

\[
\int_{x = 1/3}^{3} (10x - x^2 - 2x^2 - 3)\,dx = \int_{x = 1/3}^{3} (10x - 3x^2 - 3)\,dx
\]

\[
= \left[ 5x^2 - x^3 - 3x \right]_3^{1/3} = \frac{256}{27} \quad \text{(square units)}
\]

(c) Show (by substitution or otherwise) that

\[
\int \frac{dx}{1 + x^2} = \arctan x + C
\]

where \( C \) is an arbitrary constant. Hence by expanding \((1 + x^2)^{-1}\) as a power series in \( x \) and integrating term-by-term derive a power series for \( \arctan x \). Deduce that

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots
\]

Put \( x = \tan \theta \) so that \( \theta = \arctan x \). Then

\[
\int \frac{dx}{1 + x^2} = \int \frac{\sec^2 \theta \, d\theta}{1 + \tan^2 \theta} = \int d\theta = \theta + C
\]

Now write

\[
\arctan \theta = \int \frac{dx}{1 + x^2} = \int 1\,dx - \int x^2\,dx + \int x^4\,dx - \ldots
\]

\[
= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots
\]

where \( C = 0 \) because \( \arctan 0 = 0 \). Now put \( \theta = \pi/4 \), i.e. \( x = 1 \). Then we obtain
arctan1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... \\

as required.