CARDIFF UNIVERSITY
EXAMINATION PAPER

Academic Year: 2001/2002
Examination Period: SPRING
Examination Paper Number: CM0188
Examination Paper Title: Analytic Mathematics (I)
Duration: Two hours

Do not turn this page over until instructed to do so by the Senior Invigilator.

Structure of Examination Paper:

There are FIVE pages.
There are FOUR questions in total.
There are no appendices.

The maximum mark for the examination paper is 100% and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Students to be provided with:

The following items of stationery are to be provided:

ONE answer book.

Instructions to Students:

Answer THREE questions.

The use of translation dictionaries between English or Welsh and a foreign language bearing an appropriate departmental stamp is permitted in this examination.
1. (i) Show that if $z \in \mathbb{C}$ and

$$s_n = 1 + z + z^2 + \ldots + z^n \quad (n \geq 0)$$

then

$$s_n = \frac{1 - z^{n+1}}{1 - z} \quad \text{(provided } z \neq 1)$$

(ii) Hence show that

$$\sum_{n=1}^{\infty} z^n = \frac{1}{1 - z} \quad (|z| < 1)$$

(iii) Show that if $z = r(\cos \theta + i \sin \theta)$ where $i^2 = -1$ and $r > 0$ then for $z \neq 1$

$$\frac{1}{1 - z} = \frac{1 - r \cos \theta}{1 - 2r \cos \theta + r^2} + i \frac{r \sin \theta}{1 - 2r \cos \theta + r^2}$$

(iv) Now again putting $z = r(\cos \theta + i \sin \theta)$ in (ii) show that for any real $\theta$

$$1 + r \cos \theta + r^2 \cos 2\theta + \ldots = \frac{1 - r \cos \theta}{1 - 2r \cos \theta + r^2} \quad (|r| < 1)$$

$$r \sin \theta + r^2 \sin 2\theta + \ldots = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2} \quad (|r| < 1)$$

provided $0 \leq r < 1$. 

[5]
2. (a) (i) Starting from the power series expansion

\[ \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (|x| < 1) \]

Show that

\[ \log\left(\frac{1 + x}{1 - x}\right) = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \right) \quad (|x| < 1) \]  

(ii) Hence show how \(\log(11/9)\) may be calculated by hand and perform this calculation rounding to 4 decimal places.

(b) Solve the simultaneous equations

\[
\begin{align*}
3x + 5y &= 10 \pmod{11} \\
2x + 4y &= 5 \pmod{11}
\end{align*}
\]

in the field of integers (mod 11).

(c) Express the following complex number in the form \(a + ib\), where \(a\) and \(b\) are real numbers:

\[
\left( \frac{1}{2} + i\frac{\sqrt{3}}{2} \right)^{12}
\]

[Hint: Use De Moivre’s theorem.]
3. (a) State the integral test for convergence/divergence of the series $\sum a_n$, and illustrate the idea involved with a sketch. [5]

(b) Explain the notion of the ‘radius of convergence’ of a power series. [2]

(c) Check that the examples

(i) $a_n = 1/n$  \hspace{1cm} (n \geq 1) [4]

(ii) $a_n = 1/n^{1+\epsilon}$  \hspace{1cm} (n \geq 1, and any $\epsilon > 0$) [4]

satisfy the conditions of the integral test and discuss the conclusions when the integral test is applied to the series $\sum a_n$.

(d) Differentiate

$$\frac{\text{Exp}(x^2)}{\text{Log}(1+x)}$$ \hspace{1cm} (x > -1)

with respect to $x$ [5]
4  (a) Briefly explain by means of a diagram how the expression
\[ \int_{x=a}^{b} f(x) \, dx \]
is defined in terms of a limiting process involving upper and lower sums.  \[7\]

(b) Find the volume of a spherical cap of height \( h \) cut from a sphere of radius \( R \) (see Figure 1).  \[8\]

(c) Evaluate the definite integral
\[ \int_{x=0}^{1} \frac{dx}{\exp(x) + \exp(-x)} \]
leaving your answer in terms of \( e \).  \[5\]
Solutions

1. (i) Multiply by \( z \) we write
\[
\begin{align*}
s_n &= 1 + z + z^2 + \ldots + z^n \quad (n \geq 0) \\
zs_n &= z + z^2 + \ldots + z^n + z^{n+1}
\end{align*}
\]
Subtracting the second equation from the first we have
\[
(1 - z)s_n = 1 - z^{n+1}
\]
Hence on dividing both sides by \( 1 - z \) we have
\[
s_n = \frac{1 - z^{n+1}}{1 - z} \quad \text{(provided } z \neq 1)\]
(ii) If \( |z| < 1 \) then \( z^{n+1} \to \infty \) as \( n \to \infty \) so that
\[
\lim_{n \to \infty} s_n = \frac{1}{1 - z}
\]
But \( s_n \) is the \( n \)th partial sum of the series \( \sum z^n \), hence by definition
\[
\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \quad (|z| < 1)
\]
(iii) Observing that
\[
\frac{1}{1 - z} = \frac{1}{1 - r \cos \theta - i r \sin \theta}
\]
\[
= \frac{(1 - r \cos \theta + i r \sin \theta)}{(1 - r \cos \theta - i r \sin \theta)(1 - r \cos \theta + i r \sin \theta)}
\]
\[
= \frac{1 - r \cos \theta + i r \sin \theta}{1 - 2 r \cos \theta + r^2}
\]
we have the required result on separating into real and imaginary parts.

(iv) Putting \( z = r (\cos \theta + i \sin \theta) \) where \( r > 0 \) we have by De Moivre’s theorem
\[
z^n = r^n (\cos \theta + i \sin \theta)^n = r^n (\cos n\theta + i \sin n\theta)
\]
Hence
\[
\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} r^n \cos n\theta + i \sum_{n=0}^{\infty} r^n \sin n\theta
\]
\[
= (1 + r \cos \theta + r^2 \cos 2\theta + \ldots) + i (r \sin \theta + r^2 \sin 2\theta + \ldots)
\]
We now use parts (ii) and (iii) to obtain the required result by equating real and imaginary parts.
2. (a) (i) We have 
\[
\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots
\]
\[
\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots
\]
Hence
\[
\log\left(\frac{1 + x}{1 - x}\right) = \log(1 + x) - \log(1 - x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)
\]
for |x| < 1.

(ii) Hence taking 
\[
\frac{1 + x}{1 - x} = \frac{11}{9}
\]
\[
i.e \quad 9 + 9x = 11 - 11x \quad \text{or} \quad x = \frac{1}{10}
\]
(so that |x| < 1) we have 
\[
\log\frac{11}{9} = 2\left(0.1 + 0.333333\times10^{-3} + 0.20000\times10^{-5}\right) = 0.20070
\]
In fact \(\log 11/9 \approx 0.200671\).

(b) Multiplying the first equation by 2, the second by 3 and remembering to perform arithmetic (mod 11) we obtain 
\[
6x + 10y = 20 \equiv 9 \pmod{11}
\]
\[
6x + y = 15 \equiv 4 \pmod{11}
\]
Subtracting the second from the first we have \(9y \equiv 5 \pmod{11}\) which has the solution \(y = 3\) whence \(x = 2 \pmod{11}\).

(c) We have 
\[
\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{12} = \left(\cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)^{12}
\]
\[
= \cos 4\pi + isin 4\pi = 1
\]
3. (a) The integral test is illustrated in the following figure.

![Figure 2](image_url)

**Figure 2** Comparison of the integral and the sum.

**Theorem 15.** Suppose that \( f(x) \) is a real valued function such that \( f(x) \) is monotonic decreasing, \( f(x) \geq 0 \) for all \( x \geq 1 \) and \( \lim_{x \to \infty} f(x) = 0 \). Write \( a_k = f(k) \) \( (k \in \mathbb{N}) \), then

\[
\left\{ \sum_{k=1}^{\infty} a_k \text{ converges} \right\} \text{ if and only if } \left\{ \int_{1}^{\infty} f(x) \, dx \text{ converges} \right\}
\]

The idea should be clear from **Figure 2**

(b) Radius of convergence.

**Definition.** If a power series is convergent for \( |x| < R \) and divergent for some \( x \) with \( |x| > R \) (notice we say nothing about what happens if \( |x| = R \)) then we call \( R \) the *radius of convergence* of the power series.

Thus the radius of convergence is the greatest \( R \in \mathbb{R} \) such that the series converges for \( |x| < R \).

It is called the *radius* because for power series with complex terms the region of convergence is a circle of radius \( R \) in the complex plane.

(c) (i) Now \( f(n) = 1/n \) is strictly positive and monotonic decreasing to zero. Moreover \( \lim_{n \to \infty} 1/n = 0 \) so the integral test can be applied. We have

\[
\int_{1}^{X} \frac{dx}{x} = \left[ \log x \right]_{1}^{X} = \log X - \infty \text{ as } X \to \infty
\]

Hence the series is divergent.
CM0188

(ii) Now \( f(n) = 1/n^{1+\varepsilon} \) is strictly positive and monotonic decreasing to zero for any \( \varepsilon > 0 \). Moreover \( \lim 1/n^{1+\varepsilon} = 0 \) as \( n \to \infty \) so the integral test can be applied. We have

\[
\int_{x}^{\infty} \frac{dx}{x^{1+\varepsilon}} = \left[-\frac{1}{\varepsilon x^{\varepsilon}}\right]_{1}^{x} = \frac{1}{\varepsilon} \left(1 - \frac{1}{x^\varepsilon}\right) \to \frac{1}{\varepsilon} \text{ as } X \to \infty
\]

Hence the series is convergent.

(d) Using the product or quotient rule we have

\[
\frac{d}{dx} \left( \frac{\exp(x^2)}{\log(1 + x)} \right) = -\frac{\exp(x^2)}{(1 + x) \log(1 + x)^2} + \frac{2\exp(x^2)x}{\log(1 + x)}
\]
The area under a curve \( y = f(x) \) which is written as

\[
\int_{a}^{b} f(x) \, dx
\]

is defined by a limiting process. We divide up the interval \([a, b]\) into a finite number of sub-intervals whose maximum length is \( \delta \) (say). Next we form the upper and lower rectangles as shown in Figure 3.

For a partition into \( n \) sub-intervals with \( x_0 = a \) and \( x_n = b \) with the maximum length of the sub-intervals being \( \delta \), we write

\[
L(\delta) = \sum_{i=0}^{n-1} \left( \min_{x \in [x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i)
\]

\[
U(\delta) = \sum_{i=0}^{n-1} \left( \max_{x \in [x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i)
\]

Then

\[
L(\delta) \leq U(\delta)
\]

where \( L(\delta) \) is the sum of the areas of the lower rectangles and \( U(\delta) \) is the sum of the areas of the upper rectangles. (There is no problem in defining the area of a rectangle in the usual way.)

As \( \delta \to 0 \) one can show (under reasonable conditions on \( f \)) that \( L(\delta) \) is monotonic increasing and \( U(\delta) \) is monotonic decreasing. These functions are both bounded so that

\[
\lim_{\delta \to 0} L(\delta) \quad \text{and} \quad \lim_{\delta \to 0} U(\delta)
\]

both exist.
**Definition.** If both limits exist and are equal we define their common value to be
\[ \int_{x=a}^{b} f(x) \, dx \]

This is called the *integral* of \( f \) from \( a \) to \( b \).

[Some reasonable summary of this is sufficient]

b) Integrating the volume of circular disks of thickness \( dx \) from \( R - h \) to \( R \) we have
\[
\int_{R-h}^{R} \pi(R^2 - x^2) \, dx = \pi \left( R^2 x - \frac{x^3}{3} \right)_{R-h}^{R} \\
= \frac{2\pi R^3}{3} - \pi \left( \frac{3R^3 - 3R^2 h - R^3 + 3R^2 h - 3Rh^2 + h^3}{3} \right) \\
= \frac{\pi}{3} \left( 2R^3 - 3R^2 + R^3 + 3Rh^2 - h^3 \right) = \frac{\pi h^2}{3} (3R - h)
\]

c) Now multiplying top and bottom by \( \text{Exp}(x) \) and substituting \( t = \text{Exp}(x) \) we have
\[
\int_{x=0}^{1} \frac{\text{Exp}(x)}{\text{Exp}(2x) + 1} \, dx = \int_{t=1}^{e} \frac{dt}{t^2 + 1} = \left[ \arctan(t) \right]_{1}^{e} \\
= \arctan(e) - \frac{\pi}{4}
\]