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Asymptotic moments of near-neighbour distance distributions

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Let C be a compact convex body in $\mathbb{R}^m$ and consider a set of points selected at random from C according to some well-behaved sampling distribution. We obtain an asymptotic expression for the positive moments of the kth near-neighbour distance distribution as the number of points increases to infinity.

Keywords: near-neighbour distance distributions; moments; point processes

1. Introduction

Let $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$ be a set of M points selected uniformly at random from the unit hypercube in $\mathbb{R}^m$ and let $d_{M,k}$ denote the distance between any point of the set and its kth nearest neighbour in the set. In Percus & Martin (1998) it is shown that under periodic boundary conditions, the expected value of $d_{M,k}$ satisfies the asymptotic expression

$$\mathcal{E}(d_{M,k}) = V_m^{-1/m} \frac{\Gamma(k+1/m)}{\Gamma(k)} \frac{1}{M^{1/m}} + O\biggl(\frac{1}{M^{1+1/m}}\biggr) \quad \text{as } M \to \infty, \tag{1.1}$$

where V_m denotes the volume of the unit ball in $\mathbb{R}^m$. In this paper we prove a similar result for the αth non-negative moment $\mathcal{E}(d_{M,k}^{\alpha})$ of the kth nearest-neighbour distance distribution with the following important generalizations.

(i) The points $\boldsymbol{x}_i$ may be selected from any compact convex body C in $\mathbb{R}^m$.

(ii) The points $\boldsymbol{x}_i$ may be selected according to any well-behaved sampling density ϕ on C.

We show that, for all $0 < \rho < 1/m$,

$$\mathcal{E}(d_{M,k}^{\alpha}) = \frac{c(m,\alpha,k,\phi)}{M^{\alpha/m}} + O\biggl(\frac{1}{M^{(\alpha+1)/m-\rho}}\biggr) \quad \text{as } M \to \infty, \tag{1.2}$$

where

$$c(m,\alpha,k,\phi) = V_m^{-\alpha/m} \frac{\Gamma(k+\alpha/m)}{\Gamma(k)} \int_C \phi(\boldsymbol{x})^{1-\alpha/m} \, \mathrm{d}\boldsymbol{x} \tag{1.3}$$

is a finite constant not depending on M.

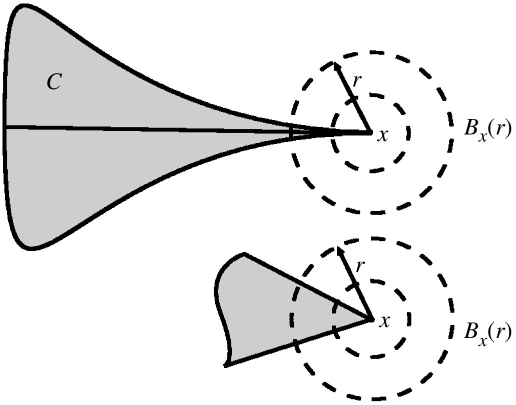

Figure 1. Condition **C.1** eliminates certain types of boundary points (top).

In what follows, μ represents the Lebesgue measure in $\mathbb{R}^m$ and ∂A denotes the boundary of a set $A \subset \mathbb{R}^m$. The ball of radius r centred at $\boldsymbol{x} \in \mathbb{R}^m$ is denoted by $B_x(r)$ and V_m denotes the Lebesgue measure of the unit ball in $\mathbb{R}^m$. Finally, for any $A \subset \mathbb{R}^m$ and $\delta > 0$, the *boundary region of width* δ is defined to be the set $A(\delta)$ consisting of those points in A which are within (Euclidean) distance δ of the boundary,

$$A(\delta) = \Big\{ \boldsymbol{x} \in A : \inf_{\boldsymbol{y} \in \partial A} |\boldsymbol{x} - \boldsymbol{y}| < \delta \Big\}. \tag{1.4}$$

2. The set C

Let C be any compact subset of $\mathbb{R}^m$ having diameter c_1 ($0 < c_1 < \infty$), and without loss of generality suppose that $\mu(C) = 1$. We first impose the following geometric conditions on C (see figures 1 and 2).

C.1 There exists some constant $c_2 > 0$ such that, for all $\boldsymbol{x} \in C$ and $0 < r < c_1$,

$$\mu(B_x(r) \cap C) > c_2 r^m, \tag{2.1}$$

i.e. at least a uniformly constant proportion of the ball $B_x(r)$ must intersect C.

C.2 There exist constants $\lambda = \lambda(C) > 0$ and $c_3 > 0$ such that, for all $0 < \delta < \lambda$,

$$\mu(C(\delta)) \leqslant c_3 \delta, \tag{2.2}$$

i.e. the measure of the boundary region must be uniformly bounded above by some constant multiple of δ.

Proposition 2.1. *Conditions* **C.1** *and* **C.2** *are satisfied by compact convex bodies in* $\mathbb{R}^m$.

Proof. Let C be a compact convex body in $\mathbb{R}^m$. To prove condition **C.1** we note that by definition a convex body has a non-empty interior, so there exist points $\boldsymbol{a}, \boldsymbol{b} \in C$ along with some $r_a, r_b > 0$ such that the balls $B_a(r_a)$ and $B_b(r_b)$ are disjoint and completely contained in C.

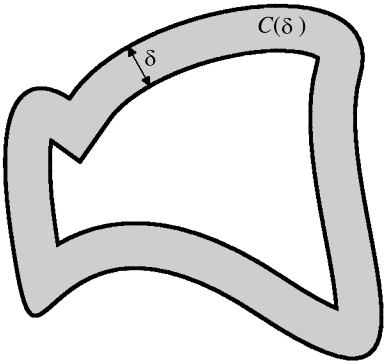

Figure 2. Condition **C.2** requires that the measure of $C(\delta)$ is bounded above by some constant multiple of δ.

First suppose that $\boldsymbol{x} \in C \setminus B_a(r_a)$ and consider the cone C_x having vertex $\boldsymbol{x}$ and base equal to the intersection of $B_a(r_a)$ with the hyperplane through $\boldsymbol{a}$ perpendicular to the line joining $\boldsymbol{x}$ and $\boldsymbol{a}$. By convexity, C_x is completely contained in C, and since $r_a > 0$ we find that C_x is of positive volume for each $\boldsymbol{x} \in C \setminus B_a(r_a)$.

Let $c_x(r) > 0$ denote the proportion of the ball $B_x(r)$ occupied by the cone C_x. As r increases from 0 to c_1, the proportion $c_x(r)$ remains constant while $r \leqslant |\boldsymbol{x} - \boldsymbol{a}|$ and then decreases monotonically for $r > |\boldsymbol{x} - \boldsymbol{a}|$. Let $c_x = c_x(c_1)$ denote the minimum value of $c_x(r)$. Since the volume of C_x is positive and the volume of the ball $B_x(c_1)$ is finite, it is clear that $c_x > 0$ and furthermore that $\mu(B_x(r) \cap C) \geqslant c_x \mu(B_x(r)) > 0$ for all $0 < r \leqslant c_1$.

Define c_a to be the minimum value of c_x over all points $\boldsymbol{x} \in C \setminus B_a(r_a)$. This corresponds to those points $\boldsymbol{x}$ which lie on the boundary of the ball $B_a(r_a)$ (for which the cones C_x are of minimum volume) and, since $B_a(r_a)$ is of positive radius, we have $c_a > 0$. Hence $\mu(B_x(r) \cap C) \geqslant c_a \mu(B_x(r)) > 0$ for all $0 < r \leqslant c_1$ and $\boldsymbol{x} \in C \setminus B_a(r_a)$.

Now suppose that $\boldsymbol{x} \in B_a(r_a)$ and define C_x and c_x as above, this time relative to the ball $B_b(r_b)$. Define c_b to be the minimum value of c_x over each $\boldsymbol{x} \in B_a(r_a)$. Since $B_a(r_a)$ and $B_b(r_b)$ are disjoint and since $B_b(r_b)$ is of positive radius, we find that $c_b > 0$ and hence $\mu(B_x(r) \cap C) \geqslant c_b \mu(B_x(r)) > 0$ for all $0 < r \leqslant c_1$ and $\boldsymbol{x} \in B_a(r_a)$.

Finally, letting $c = \min\{c_a, c_b\} > 0$ we find that for all $0 < r \leqslant c_1$ and for every $\boldsymbol{x} \in C$,

$$\mu(B_x(r) \cap C) \geqslant c\mu(B_x(r)) \geqslant cV_m r^m > 0, \tag{2.3}$$

where V_m is the volume of the unit ball in $\mathbb{R}^m$ and hence condition **C.1** is satisfied with $c_2 = cV_m > 0$.

Next we show that C satisfies condition **C.2**. By definition, C has a non-empty interior, and we may suppose (without loss of generality) that the origin $\boldsymbol{0} \in \operatorname{int} C$. Let $B_0(\lambda)$ denote the ball of maximal radius $\lambda > 0$ centred at the origin and contained in C.

Let $0 < \delta < \lambda$ and suppose that $\boldsymbol{x} \in C(\delta)$. Let $\boldsymbol{y}$ be a boundary point of C such that the distance from $\boldsymbol{x}$ to $\boldsymbol{y}$ is strictly less than δ (this exists by definition of $C(\delta)$) and define $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$, where $|\boldsymbol{z}| < \delta$.

Let us write $-\boldsymbol{z} = (\delta/\lambda)\boldsymbol{a}$, where $\boldsymbol{a} \in \mathbb{R}^m$. Then $\boldsymbol{a} = (\lambda/\delta)(-\boldsymbol{z})$ and $|\boldsymbol{a}| = (\lambda/\delta)|\boldsymbol{z}|$. By definition, $|\boldsymbol{z}| < \delta$ so we have $|\boldsymbol{a}| < \lambda$ and hence $\boldsymbol{a} \in \operatorname{int} B_0(\lambda)$. Since $B_0(\lambda) \subseteq C$, this implies that $\boldsymbol{a} \in \operatorname{int} C$ and therefore $-\boldsymbol{z} \in \operatorname{int}(\delta/\lambda)C$.

Now suppose that $\boldsymbol{x} \in (1 - \delta/\lambda)C$. Then $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{z}$ may be expressed as $\boldsymbol{y} = (1 - \delta/\lambda)\boldsymbol{b} + (\delta/\lambda)\boldsymbol{a}$ for some $\boldsymbol{b} \in C$ and $\boldsymbol{a} \in \operatorname{int} C$. Since $0 < \delta < \lambda$, we have $0 < \delta/\lambda < 1$ and $0 < 1 - \delta/\lambda < 1$ and, since C is convex and $\boldsymbol{a} \in \operatorname{int} C$, this implies that $\boldsymbol{y} \in \operatorname{int} C$, contradicting the fact that $\boldsymbol{y}$ is a boundary point of C.

Hence $\boldsymbol{x} \notin (1-\delta/\lambda)C$ so the sets $C(\delta)$ and $(1-\delta/\lambda)C$ are disjoint. $C(\delta)$ is therefore contained in $C \setminus (1 - \delta/\lambda)C$ so

$$\mu(C(\delta)) \leqslant \mu(C) - \mu((1 - \delta/\lambda)C). \tag{2.4}$$

Since $\mu((1 - \delta/\lambda)C) = (1 - \delta/\lambda)^m \mu(C)$ and $\mu(C) = 1$, we have

$$\mu(C(\delta)) \leqslant 1 - (1 - \delta/\lambda)^m. \tag{2.5}$$

For all $0 < \delta < \lambda$ we may therefore conclude that $\mu(C(\delta)) \leqslant c_3\delta$ for some constant $c_3 > 0$, as required. ■

3. The sampling distribution *Φ*

Suppose that points $\boldsymbol{x}$ are selected at random from C according to a sampling distribution Φ. We restrict our attention to those distributions Φ for which the corresponding density function ϕ satisfies the following conditions.

P.1 ϕ is continuous on C.

P.2 ϕ has bounded partial derivatives at each point of C.

P.3 $\phi(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in C$.

Since C is compact and since ϕ is continuous and strictly positive on C, it is easily shown that there exist constants a_1, a_2 such that

$$0 < a_1 < \phi(\boldsymbol{x}) < a_2 < \infty \quad \text{for all } \boldsymbol{x} \in C. \tag{3.1}$$

Let $\omega_x(r)$ denote the probability measure induced by the sampling distribution Φ on the spherical neighbourhoods of C,

$$\omega_x(r) = \boldsymbol{P}(B_x(r)) = \int_{B_x(r)\cap C} \phi(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}. \tag{3.2}$$

Then $\omega_x(r)$ is the probability that a point selected from C is selected from the ball $B_x(r)$. We remark that for all $\boldsymbol{x} \in C$ and all $0 \leqslant r \leqslant c_1$, by (3.1) and condition **C.1**,

$$\omega_x(r) \geqslant a_1\mu(B_x(r) \cap C) \geqslant a_1 c_2 r^m \tag{3.3}$$

and we say that $\omega_x(r)$ satisfies a *positive density* condition on C.

4. The nearest-neighbour distance distribution

Suppose that M points $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\}$ are selected at random from C according to the sampling distribution Φ. For any fixed point $\boldsymbol{x} \in C$ the *kth nearest neighbour* of $\boldsymbol{x}$ in the set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\}$ is defined to be that point $\boldsymbol{x}_j$ with the property that exactly $k-1$ points of the set are closer to $\boldsymbol{x}$ than $\boldsymbol{x}_j$ is to $\boldsymbol{x}$. We note that, since $\mu(C) = 1$ and $\phi(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in C$, the kth nearest neighbour of $\boldsymbol{x}$ is uniquely defined with probability 1.

Let $d_{M,k}(\boldsymbol{x})$ denote the distance from $\boldsymbol{x}$ to its kth nearest neighbour in the set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\}$. Then $d_{M,k}(\boldsymbol{x})$ is a random variable taking values in the range $[0, c_1]$ and defined on the space of all samples of size M selected from C. Its distribution function is defined by

$$q_x(r) = \boldsymbol{P}(d_{M,k}(\boldsymbol{x}) \leqslant r) = q_x(M, k, r). \tag{4.1}$$

Following Percus & Martin (1998) we derive the corresponding density function as follows.

Lemma 4.1. *For fixed $\boldsymbol{x} \in C$, the probability density function of the random variable $d_{M,k}(\boldsymbol{x})$ is given by*

$$\mathrm{d}q_x(r) = k\binom{M}{k}\omega_x(r)^{k-1}(1-\omega_x(r))^{M-k}\,\mathrm{d}\omega_x(r). \tag{4.2}$$

Proof. Let $\epsilon > 0$ and consider

$$q_x(r+\epsilon) - q_x(r) = \boldsymbol{P}(r \leqslant d_{M,k}(\boldsymbol{x}) \leqslant r+\epsilon). \tag{4.3}$$

Since ϕ is continuous on C, for ϵ sufficiently small we may suppose that the kth nearest neighbour of $\boldsymbol{x}$ is the only point lying in the spherical shell of radius r and width $\epsilon > 0$ centred at $\boldsymbol{x}$. In this case, as illustrated in figure 3, we must have

(i) $k-1$ points in the ball $B_x(r)$, each selected with probability $\omega_x(r)$;

(ii) exactly one of the remaining $M-k+1$ points in the shell $B_x(r+\epsilon) \setminus B_x(r)$, selected with probability $\omega_x(r+\epsilon) - \omega_x(r)$;

(iii) the remaining $M-k$ points in the region $C \setminus B_x(r+\epsilon)$, each selected with probability $1-\omega_x(r+\epsilon)$.

Using elementary combinatorial arguments we have

$$q(\boldsymbol{x}, r+\epsilon) - q(\boldsymbol{x}, r) = k\binom{M}{k}\omega_x(r)^{k-1}(1-\omega_x(r+\epsilon))^{M-k}(\omega_x(r+\epsilon) - \omega_x(r)), \tag{4.4}$$

and letting $\epsilon \to 0$ we obtain (4.2), as required. ■

5. Moments of the near-neighbour distance distributions

Consider the αth moment about zero of the kth nearest-neighbour distance $d_{M,k}(\boldsymbol{x})$, defined by

$$\mathcal{E}(d_{M,k}^{\alpha}(\boldsymbol{x})) = \int_0^{c_1} r^{\alpha}\,\mathrm{d}q_x(r). \tag{5.1}$$

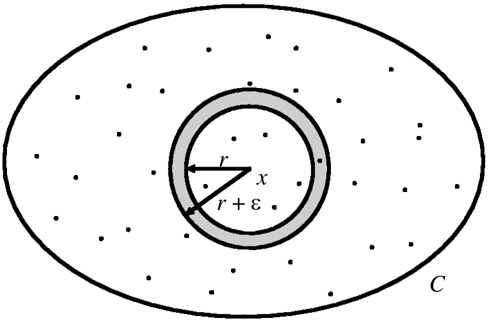

Figure 3. Exactly one point falls in the shaded region $B_x(r+\epsilon) \setminus B_x(r)$.

By lemma 4.1 we can write this as

$$\mathcal{E}(d^{\alpha}_{M,k}(\boldsymbol{x})) = k\binom{M}{k}\int_{r=0}^{c_1} r^{\alpha}\omega_x(r)^{k-1}(1-\omega_x(r))^{M-k}\,\mathrm{d}\omega_x(r). \tag{5.2}$$

Since $\omega_x(r)$ satisfies a positive density condition on C, and since C is convex, we see that $\omega_x(r)$ is strictly monotonic increasing for $0 \leqslant r \leqslant r_0$ for some $r_0 > 0$ and $\omega_x(r) = 1$ for $r_0 \leqslant r \leqslant c_1$. Writing $r = h(\omega)$ for the inverse function where it exists, we change the variable of integration in (5.2) to obtain

$$\mathcal{E}(d^{\alpha}_{M,k}(\boldsymbol{x})) = k\binom{M}{k}\int_{0}^{1} h(\omega_x)^{\alpha}\omega_x^{k-1}(1-\omega_x)^{M-k}\,\mathrm{d}\omega_x, \tag{5.3}$$

where we have used the fact that $\omega_x(0) = 0$ and $\omega_x(r_0) = 1$.

Thus the expectation $\mathcal{E}(d^{\alpha}_{M,k}(\boldsymbol{x}))$ is defined as an integral over the probability measure $\omega_x(r)$ of the spherical neighbourhoods of $\boldsymbol{x}$. We aim to find an asymptotic expression for (5.3) in terms of the number of points M, as $M \to \infty$. In order to achieve this we need the following technical results, which we state without proof.

Lemma 5.1. *For any fixed* $\sigma > 0$,

$$\frac{\Gamma(N)}{\Gamma(N+\sigma)} = \frac{1}{N^{\sigma}}\left(1 + O\left(\frac{1}{N}\right)\right) \quad \text{as } N \to \infty. \tag{5.4}$$

Lemma 5.2 (the exponential convergence lemma). *Let* $c > 0$ *and* $0 < \sigma < 1$ *be constants. Then for every* $\beta > 0$

$$\left(1 - \frac{c}{N^{\sigma}}\right)^{N} = O\left(\frac{1}{N^{\beta}}\right) \quad \text{as } N \to \infty. \tag{5.5}$$

Using these results, we first show that any ball whose radius does not shrink to zero sufficiently rapidly (relative to M) can be ignored in the limit as $M \to \infty$. More precisely, for any $0 < \rho < 1/m$ we define

$$\delta = \frac{1}{M^{1/m-\rho}} \tag{5.6}$$

and show that any ball of radius $r > \delta$ or equivalently of probability measure $\omega_x > \omega_x(\delta)$ can be neglected in the asymptotic limit as $M \to \infty$.

Lemma 5.3. *For every* $\beta > 0$,

$$k\binom{M}{k}\int_{\omega_x(\delta)}^{1} h(\omega)^{\alpha}\omega^{k-1}(1-\omega)^{M-k}\,\mathrm{d}\omega = O\biggl(\frac{1}{M^{\beta}}\biggr) \quad \textit{as } M \to \infty. \tag{5.7}$$

Proof. Let

$$I(\delta) = k\binom{M}{k}\int_{\omega_x(\delta)}^{1} h(\omega)^{\alpha}\omega^{k-1}(1-\omega)^{M-k}\,\mathrm{d}\omega. \tag{5.8}$$

By (3.3), $\omega_x(\delta) \geqslant a_1c_2\delta^m$ so for each ω in the range $\omega_x(\delta) \leqslant \omega \leqslant 1$ we have $1-\omega \leqslant 1 - a_1c_2\delta^m$. Clearly, $h(\omega) \leqslant c_1$ and $|\omega| \leqslant 1$ so

$$I(\delta) \leqslant c_1^{\alpha}k\binom{M}{k}(1-a_1c_2\delta^m)^{M-k}. \tag{5.9}$$

By lemma 5.1,

$$k\binom{M}{k} = \frac{\Gamma(M+1)}{\Gamma(M+1-k)\Gamma(k)} = O((M+1)^k) = O(M^k) \quad \text{as } M \to \infty. \tag{5.10}$$

Furthermore, $0 < \rho < 1/m$ implies that $0 < 1 - m\rho < 1$ so $\delta^m = o(1)$ and hence $(1 - a_1c_2\delta^m)^{-k} = O(1)$ as $M \to \infty$. Since $c_1 < \infty$ and α is fixed, we thus have

$$I(\delta) \leqslant O(M^k)(1-a_1c_2\delta^m)^M \quad \text{as } M \to \infty. \tag{5.11}$$

Substituting for δ we get

$$I(\delta) \leqslant O(M^k)\biggl(1 - \frac{a_1c_2}{M^{1-m\rho}}\biggr)^M \quad \text{as } M \to \infty \tag{5.12}$$

and the result follows by lemma 5.2. ■

In evaluating (5.3) we may therefore restrict our attention to those balls in the neighbourhood of $\boldsymbol{x}$ having probability measure in the range $[0, \omega_x(\delta)]$

(*a*) *Main theorem*

Theorem 5.4. *Let* C *be a compact convex body in* $\mathbb{R}^m$ *with* $\mu(C) = 1$ *and let* $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$ *be a set of points selected independently at random from* C *according to the sampling distribution* Φ *whose density function* ϕ *satisfies* **P.1**–**P.3**. *Let* $d_{M,k}$ *denote the distance between any point* $\boldsymbol{x}_i$ *and its* k*th nearest neighbour in the set* $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$. *Then for all* $0 < \rho < 1/m$ *and integer* $\alpha \geqslant 0$,

$$\mathcal{E}(d_{M,k}^{\alpha}) = \frac{c(m,\alpha,k,\phi)}{M^{\alpha/m}} + O\biggl(\frac{1}{M^{(\alpha+1)/m-\rho}}\biggr) \quad \textit{as } M \to \infty, \tag{5.13}$$

where

$$c(m,\alpha,k,\phi) = V_m^{-\alpha/m}\frac{\Gamma(k+\alpha/m)}{\Gamma(k)}\int_C \phi(\boldsymbol{x})^{1-\alpha/m}\,\mathrm{d}\boldsymbol{x} \tag{5.14}$$

is a constant not depending on M *and* V_m *is the volume of the unit ball in* $\mathbb{R}^m$.

Proof. Fix some point $\boldsymbol{x}$ of the sample $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$ and define $d_{M,k}(\boldsymbol{x})$ to be the distance from $\boldsymbol{x}$ to its kth nearest neighbour. By (5.3) applied to a set of $M-1$ points we have

$$\mathcal{E}(d^{\alpha}_{M,k}(\boldsymbol{x})) = C_{M,k} \int_0^1 h(\omega_x)^{\alpha} \omega_x^{k-1} (1-\omega_x)^{M-k-1} \, \mathrm{d}\omega_x, \tag{5.15}$$

where $h(\omega_x)$ is the radius of the ball centred at $\boldsymbol{x}$ of probability measure ω and

$$C_{M,k} = \frac{\Gamma(M)}{\Gamma(M-k)\Gamma(k)}. \tag{5.16}$$

The theorem is proved by computing the expected value of (5.15) over all points $\boldsymbol{x}$ selected from C according to Φ, given by

$$\mathcal{E}(d^{\alpha}_{M,k}) = \int_C \mathcal{E}(d^{\alpha}_{M,k}(\boldsymbol{x}))\phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{5.17}$$

Since C is convex it is connected and since ϕ is continuous on C, we can apply the first mean-value theorem of the integral calculus to ϕ. Hence by (3.2) there exists a point $\boldsymbol{\xi}_1 \in B_x(r) \cap C$ such that

$$\omega_x(r) = \phi(\boldsymbol{\xi}_1)\mu(B_x(r) \cap C). \tag{5.18}$$

Furthermore, since ϕ is differentiable at every point of C, we can also apply the first mean-value theorem of the differential calculus to ϕ. Hence there exists a point $\boldsymbol{\xi}_2$ on the line segment joining $\boldsymbol{x}$ and $\boldsymbol{\xi}_1$ such that

$$\phi(\boldsymbol{\xi}_1) = \phi(\boldsymbol{x}) + (\boldsymbol{x} - \boldsymbol{\xi}_1)\phi'(\boldsymbol{\xi}_2) \tag{5.19}$$

and since C is convex, $\boldsymbol{\xi}_2$ is contained in $B_x(r) \cap C$. For $0 < \rho < 1/m$ we define δ as in (5.6), and by lemma 5.3 we may suppose without loss of generality that $r \in [0, \delta]$. Since $\boldsymbol{\xi}_1 \in B_x(r) \cap C$, we may therefore assume that $|\boldsymbol{x} - \boldsymbol{\xi}_1| \leqslant \delta$ and write

$$|\boldsymbol{x} - \boldsymbol{\xi}_1| = O(\delta) \quad \text{as } M \to \infty. \tag{5.20}$$

By hypothesis all partial derivatives of ϕ are bounded at each point of C and since $\boldsymbol{\xi}_2 \in C$, by (5.19) and (5.20) we have

$$\phi(\boldsymbol{\xi}_1) = \phi(\boldsymbol{x}) + O(\delta) \quad \text{as } M \to \infty \tag{5.21}$$

and hence by (5.18),

$$\omega_x(r) = (\phi(\boldsymbol{x}) + O(\delta))\mu(B_x(r) \cap C) \quad \text{as } M \to \infty. \tag{5.22}$$

Let $B = C(\delta)$ denote the boundary region of width δ as defined in (1.4) and let $A = C \setminus B$. We define the conditional expectations $\mathcal{E}_A(\boldsymbol{x}) = \mathcal{E}(d^{\alpha}_{M,k}(\boldsymbol{x}) \mid \boldsymbol{x} \in A)$ and $\mathcal{E}_B(\boldsymbol{x}) = \mathcal{E}(d^{\alpha}_{M,k}(\boldsymbol{x}) \mid \boldsymbol{x} \in B)$ so that

$$\mathcal{E}(d^{\alpha}_{M,k}) = \int_A \mathcal{E}_A(\boldsymbol{x})\phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_B \mathcal{E}_B(\boldsymbol{x})\phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{5.23}$$

Case 1: $\boldsymbol{x} \in \boldsymbol{A}$. Since $\boldsymbol{x}$ is at least a distance δ from the boundary of C, by lemma 5.3 we can assume (without loss of generality) that the ball $B_x(r)$ is completely contained in C. Thus $\mu(B_x(r) \cap C) = V_m r^m$ and hence by (5.22)

$$\omega_x = (\phi(\boldsymbol{x}) + O(\delta))V_m r^m \quad \text{as } M \to \infty. \tag{5.24}$$

Since $\phi(\boldsymbol{x}) \geqslant a_1 > 0$ for all $\boldsymbol{x} \in C$, we have

$$r^m = \frac{\omega_x}{V_m \phi(\boldsymbol{x})}\left(1 + \frac{O(\delta)}{\phi(\boldsymbol{x})}\right)^{-1} = \frac{\omega_x}{V_m \phi(\boldsymbol{x})}(1 + O(\delta)) \tag{5.25}$$

as $M \to \infty$, so the inverse function $r = h(\omega_x)$ is given by

$$h(\omega_x) = (V_m \phi(\boldsymbol{x}))^{-1/m} \omega_x^{1/m} (1 + O(\delta)) \quad \text{as } M \to \infty, \tag{5.26}$$

and, since $\alpha \geqslant 0$ is fixed, we have $(1 + O(\delta))^\alpha = (1 + O(\delta))$ and hence

$$h(\omega_x)^\alpha = (V_m \phi(\boldsymbol{x}))^{-\alpha/m} \omega_x^{\alpha/m} (1 + O(\delta)) \quad \text{as } M \to \infty. \tag{5.27}$$

Substituting this into (5.15) we obtain

$$\mathcal{E}_A(\boldsymbol{x}) = (V_m \phi(\boldsymbol{x}))^{-\alpha/m} (1 + O(\delta)) I_{M,k} \quad \text{as } M \to \infty, \tag{5.28}$$

where

$$I_M(k) = C_{M,k} \int_0^1 \omega_x^{k+\alpha/m-1} (1 - \omega_x)^{M-k-1} \, \mathrm{d}\omega_x. \tag{5.29}$$

The integral in (5.29) is simply the Beta function $B(a, b)$ with parameters $a = k + \alpha/m$ and $b = M - k$, given by

$$B(k + \alpha/m, M - k) = \frac{\mathit{\Gamma}(k + \alpha/m)\mathit{\Gamma}(M - k)}{\mathit{\Gamma}(M + \alpha/m)}, \tag{5.30}$$

and substituting for $C_{M,k}$ from (5.16) we get

$$I_M(k) = \frac{\mathit{\Gamma}(k + \alpha/m)}{\mathit{\Gamma}(k)} \frac{\mathit{\Gamma}(M)}{\mathit{\Gamma}(M + \alpha/m)}. \tag{5.31}$$

By lemma 5.1 we obtain

$$I_{M,k} = \frac{\mathit{\Gamma}(k + \alpha/m)}{\mathit{\Gamma}(k)} \frac{1}{M^{\alpha/m}} \left(1 + O\left(\frac{1}{M}\right)\right) \quad \text{as } M \to \infty, \tag{5.32}$$

and by definition, $1/M = o(\delta)$ as $M \to \infty$ so that

$$\mathcal{E}_A(\boldsymbol{x}) = (V_m \phi(\boldsymbol{x}))^{-\alpha/m} \frac{\mathit{\Gamma}(k + \alpha/m)}{\mathit{\Gamma}(k)} \frac{1}{M^{\alpha/m}} (1 + O(\delta)) \quad \text{as } M \to \infty. \tag{5.33}$$

Hence

$$\int_A \mathcal{E}_A(\boldsymbol{x}) \phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \frac{c'(m, \alpha, k, \phi)}{M^{\alpha/m}} (1 + O(\delta)) \quad \text{as } M \to \infty, \tag{5.34}$$

where

$$c'(m, \alpha, k, \phi) = V_m^{-\alpha/m} \frac{\mathit{\Gamma}(k + \alpha/m)}{\mathit{\Gamma}(k)} \int_A \phi(\boldsymbol{x})^{1-\alpha/m} \, \mathrm{d}\boldsymbol{x}. \tag{5.35}$$

In fact, the error incurred in replacing the integral in (5.35) by the equivalent integral over C is at most of order $O(\delta)$ as $M \to \infty$. To see this we note that, since $0 < a_1 \leqslant \phi(\boldsymbol{x}) \leqslant a_2 < \infty$ for each $\boldsymbol{x} \in C$, then $|\phi(\boldsymbol{x})^{1-\alpha/m}| \leqslant a_3 < \infty$, where $a_3 = \max\{1/a_1^{1-\alpha/m}, a_2^{1-\alpha/m}\}$. Furthermore, by (5.6) we see that $\delta \to 0$ as $M \to \infty$, so by condition **C.2** there exists some constant $c_3 > 0$ such that $\mu(B) \leqslant c_3\delta$ for sufficiently large M. Hence

$$\int_B \phi(\boldsymbol{x})^{1-\alpha/m}\,\mathrm{d}\boldsymbol{x} \leqslant a_3 c_3 \delta = O(\delta) \quad \text{as } M \to \infty, \tag{5.36}$$

and, since $C = A \cup B$ is a disjoint union, we obtain

$$\int_A \phi(\boldsymbol{x})^{1-\alpha/m}\,\mathrm{d}\boldsymbol{x} = \int_C \phi(\boldsymbol{x})^{1-\alpha/m}\,\mathrm{d}\boldsymbol{x} + O(\delta) \quad \text{as } M \to \infty. \tag{5.37}$$

Thus, by (5.34) and (5.35) we have

$$\int_A \mathcal{E}_A(\boldsymbol{x})\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \frac{c(m,\alpha,k,\phi)}{M^{\alpha/m}} + O\biggl(\frac{\delta}{M^{\alpha/m}}\biggr) \quad \text{as } M \to \infty, \tag{5.38}$$

where

$$c(m,\alpha,k,\phi) = V_m^{-\alpha/m}\frac{\Gamma(k+\alpha/m)}{\Gamma(k)}\int_C \phi(\boldsymbol{x})^{1-\alpha/m}\,\mathrm{d}\boldsymbol{x}. \tag{5.39}$$

Case 2: $\boldsymbol{x} \in \boldsymbol{B}$. In this case the ball $B_x(r)$ is not necessarily contained in C. Condition **C.1** states that for all $r > 0$ there exists some constant $c_2 > 0$ such that $\mu(B_x(r) \cap C) \geqslant c_2 r^m$ and hence

$$\omega_x \geqslant (\phi(\boldsymbol{x}) + O(\delta))c_2 r^m. \tag{5.40}$$

Proceeding as in (5.25) of case (1) we obtain an upper bound for the inverse function $r = h(\omega_x)$ given by

$$h(\omega_x) \leqslant (c_2\phi(\boldsymbol{x}))^{-1/m}\omega_x^{1/m}(1 + O(\delta)) \quad \text{as } M \to \infty, \tag{5.41}$$

from which we conclude that

$$\mathcal{E}_B(\boldsymbol{x}) = O\biggl(\frac{1}{M^{\alpha/m}}\biggr) \quad \text{as } M \to \infty \tag{5.42}$$

and thus

$$\int_B \mathcal{E}_B(\boldsymbol{x})\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = O\biggl(\frac{1}{M^{\alpha/m}}\biggr)\int_B \phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{as } M \to \infty. \tag{5.43}$$

Since $|\phi(\boldsymbol{x})| \leqslant a_2 < \infty$ for each $\boldsymbol{x} \in B$, and by condition **C.2** as before, we have

$$\int_B \phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \leqslant a_2 c_3 \delta = O(\delta) \quad \text{as } M \to \infty \tag{5.44}$$

and hence

$$\int_B \mathcal{E}_B(\boldsymbol{x})\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = O\biggl(\frac{\delta}{M^{\alpha/m}}\biggr) \quad \text{as } M \to \infty. \tag{5.45}$$

From (5.23), (5.38) and (5.45) we thus obtain

$$\mathcal{E}(d_{M,k}^{\alpha}) = \frac{c(m,\alpha,k,\phi)}{M^{\alpha/m}} + O\biggl(\frac{\delta}{M^{\alpha/m}}\biggr) \quad \text{as } M \to \infty, \tag{5.46}$$

and, substituting for δ from (5.6), the result follows. ■

6. Discussion

(*a*) *A uniform sampling distribution*

Suppose now that the points $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$ are selected according to a *uniform* sampling distribution. Since $\mu(C) = 1$ in this case, we must have $\phi(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in C$ and the constant (5.14) reduces to

$$c(m, \alpha, k, \phi) = V_m^{-\alpha/m} \frac{\Gamma(k + \alpha/m)}{\Gamma(k)}, \tag{6.1}$$

which agrees with the constant of (1.1) obtained by Percus & Martin (1998) on a torus in the case $\alpha = 1$. Note also that the convexity condition on C may be relaxed in the case of a uniform sampling distribution, i.e. we only need conditions **C.1** and **C.2** to be satisfied. This is because $\omega_x(r) = \mu(B_x(r) \cap C)$ for each $\boldsymbol{x} \in C$ for the uniform distribution and the application of the mean-value theorems (equations (5.18) and (5.19)) in the proof of theorem 5.4 are unnecessary.

(*b*) *A law of large numbers for kth nearest-neighbour distance*

For the points $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$ let

$$\Delta_M(k) = \frac{1}{M} \sum_{i=1}^{M} d_{M,k}(\boldsymbol{x}_i) \tag{6.2}$$

be the empirical mean distance between the kth nearest neighbours in the set. In another paper (Evans & Jones 2002) we show that bounded functions of a point and its kth nearest neighbour satisfy a weak law of large numbers with explicit bounds. Applied to (6.2) this means that, for all $\kappa > 0$,

$$\Delta_M(k) = \mathcal{E}(\Delta_M(k)) + O\left(\frac{\mathcal{E}(d_{M,k}^2)^{1/4}}{M^{1/2-\kappa}}\right) \quad \text{in probability as } M \to \infty, \tag{6.3}$$

so by theorem 5.4 it follows that

$$\Delta_M(k) = \mathcal{E}(d_{M,k}) + O\left(\frac{1}{M^{1/2+1/(2m)-\kappa}}\right) \quad \text{in probability as } M \to \infty. \tag{6.4}$$

Hence for all $m \geqslant 2$ the empirical mean $\Delta_M(k)$ converges in probability to the distribution mean $\mathcal{E}(d_{M,k})$ as $M \to \infty$.

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