An Empirical Evaluation of Geometric Subjective Logic Operators

Federico Cerutti, Alice Toniolo, Nir Oren, and Timothy J. Norman

University of Aberdeen
School of Natural and Computing Science
King's College
AB24 3UE, Aberdeen, UK
f.cerutti@abdn.ac.uk
alice.toniolo@abdn.ac.uk
n.oren@abdn.ac.uk
t.j.norman@abdn.ac.uk

Abstract. Computational trust mechanisms aim to produce a trust rating from both direct and indirect information about agents behaviour. Jøsang's Subjective Logic has been widely adopted as the core of such systems via its fusion and discount operators. Recently we proposed an operator for discounting opinions based on geometrical properties, and, continuing this line of investigation, this paper describes a new geometry based fusion operator. We evaluate this fusion operator together with our geometric discount operator in the context of a trust system, and show that our operators outperform those originally described by Jøsang. A core advantage of our work is that these operators can be used without modifying the remainder of the trust and reputation system.

1 Introduction

Trust forms the backbone of human societies, improving system robustness by restricting the actions of untrusted entities and the use of untrusted information. Within the context of multi-agent systems [1], the problem of how to determine the degree of trustworthiness to assign to other agents has received great attention in literature. This level of trustworthiness is utilised when selecting partners for interactions; distrusted agents will rarely be interacted with, reducing their influence over the system.

Trust mechanisms aim to compute a level of trust based on direct and second-hand interactions between agents. The latter, commonly referred to as *reputation* information, is obtained from other agents which have interacted with the agent whose trustworthiness is being computed. Aspects of such systems that have been examined include how to minimise the damage caused by collusion between agents [2], the nature of reputation information [3], and examining trust in specific contexts and agent interaction configurations [4].

In this paper we propose an innovative way to discount and combine opinions in Jøsang's Subjective Logic [5]. This is part of an ongoing work started with [6], where the first of the two operators used in this paper was introduced. Moreover, we evaluate our proposed operators via an experiment, comparing the effectiveness of Jøsang's operators and those introduced in this work. This experiment indicates that our operators generally compute reputation opinion closer to the ground truth than Jøsang's.

In the next section we provide a brief overview of Jøsang's Subjective Logic (hereafter abbreviated SL). Section 3 recalls the main contributions from [6] and introduces the second new operator needed for combining reputation and trustworthiness opinions. Then, Sect. 4 describes the experiment we designed and the results obtained. Finally Sect. 5 concludes the paper.

2 Background and Motivations

The terms *trust*, *trustworthiness*, and *reputation* often have different meanings in different approaches. It is beyond the scope of this paper to investigate these meanings; the interested reader is referred to [7, 8] for an overview.

For the purpose of this paper, we consider the notion of *trustworthiness* as the property of an agent we are connected with and this property represents the willingness of the agent to share information in a trustworthy manner. Moreover, the *reputation* is a property of an agent we are not connected with, and this property represents the subjective view of its trustworthiness obtained from an agent with which we can directly communicate.

Following [9] we express both the *degree of trustworthiness* and the *degree of reputation* using Subjective Logic (SL). This formalism extends probability theory by expressing uncertainty about the probability values themselves, which makes it useful for representing trust degrees. We now proceed to provide a brief overview of SL mainly based on [5].

Like Dempster-Shafer evidence theory [10, 11], SL operates on a *frame of discernment*, denoted by Θ. A frame of discernment contains the set of possible system states, only one of which represents the actual system state. These are referred to as atomic, or primitive, system states. The powerset of Θ, denoted by 2^{Θ}, consists of all possible unions of primitive states. A non-primitive state may contain other states within it. These are referred to as substates of the state.

Definition 1. *Given a frame of discernment Θ, we can associate a belief mass assignment $m_{\Theta}(x)$ with each substate $x \in 2^{\Theta}$ such that*

1. $m_{\Theta}(x) \geq 0$
2. $m_{\Theta}(\emptyset) = 0$
3. $\sum_{x \in 2^{\Theta}} m_{\Theta}(x) = 1$

For a substate x, $m_{\Theta}(x)$ is its *belief mass*.

Belief mass is an unwieldy concept to work with. When we speak of belief in a certain state, we refer not only to the belief mass in the state, but also to the belief masses of the state's substates. Similarly, when we speak about disbelief, that is, the total belief that a state is not true, we need to take substates into account. Finally, SL also introduces the concept of uncertainty, that is, the amount of belief that might be in a superstate or a partially overlapping state. these concepts can be formalised as follows.

Definition 2. *Given a frame of discernment Θ and a belief mass assignment m_Θ on Θ, we define the belief function for a state x as*

$$b(x) = \sum_{y \subseteq x} m_\Theta(y) \text{ where } x, y \in 2^\Theta$$

The disbelief function as

$$d(x) = \sum_{y \cap x = \emptyset} m_\Theta(y) \text{ where } x, y \in 2^\Theta$$

And the uncertainty function as

$$u(x) = \sum_{\substack{y \cap x \neq \emptyset \\ y \not\subseteq x}} m_\Theta(y) \text{ where } x, y \in 2^\Theta$$

These functions have two important properties. First, they all range between zero and one. Second, they always sum to one, meaning that it is possible to deduce the value of one function given the other two.

Boolean logic operators have SL equivalents. It makes sense to use these equivalent operators in frames of discernment containing a state and (some form of) the state's negation. A *focused frame of discernment* is a binary frame of discernment containing a state and its complement.

Definition 3. *Given $x \in 2^\Theta$, the frame of discernment denoted by $\tilde{\Theta}^x$, which contains two atomic states, x and $\neg x$, where $\neg x$ is the complement of x in Θ, is the focused frame of discernment with focus on x.*

Let $\tilde{\Theta}^x$ be the focused frame of discernment with focus on x of Θ. Given a belief mass assignment m_Θ and the belief, disbelief and uncertainty functions for x ($b(x)$, $d(x)$ and $u(x)$ respectively), the focused belief mass assignment, $m_{\tilde{\Theta}^x}$ on $\tilde{\Theta}^x$ is defined as

$$m_{\tilde{\Theta}^x}(x) = b(x)$$
$$m_{\tilde{\Theta}^x}(\neg x) = d(x)$$
$$m_{\tilde{\Theta}^x}(\tilde{\Theta}^x) = u(x)$$

The focused relative atomicity of x (which approximates the role of a prior probability distribution within probability theory, weighting the likelihood of some outcomes over others) is defined as

$$a_{\tilde{\Theta}^x}(x/\Theta) = [E(x) - b(x)]/u(x)$$

For convenience, the focused relative atomicity of x is often abbreviated $A_{\tilde{\Theta}^x}(x)$.

An opinion consists of the belief, disbelief, uncertainty and relative atomicity as computed over a focused frame of discernment.

Definition 4. *Given a focused frame of discernment Θ containing x and its complement $\neg x$, and assuming a belief mass assignment m_Θ with belief, disbelief, uncertainty and relative atomicity functions on x in Θ of b(x),d(x),u(x) and a(x), we define an* opinion *over x, written ω_x as*

$$\omega_x \equiv \langle b(x), d(x), u(x), a(x) \rangle$$

For compactness, Jøsang also denotes the various functions as b_x,d_x,u_x and a_x in place, and we will follow his notation. Furthermore, given a fixed a_x, an opinion ω can be denoted as a $\langle b_x, d_x, u_x \rangle$ triple.

Given opinions about two propositions from different frames of discernment, it is possible to combine them in various ways using operators introduced, above all, in [5, 12–15]. In this work we concentrate on Jøsang's *discount* and *fusion* operators, which we review next.

Definition 5 (Former Def. 5 of [14]). *Let A, B be two agents where A's opinion about B's recommendations is expressed as $\omega_B^A = \langle b_B^A, d_B^A, u_B^A, a_B^A \rangle$ and let x be a proposition where B's opinion about x (e.g. the degree of trustworthiness of a third agent [ed.]) is recommended to A with the opinion $\omega_X^B = \langle b_x^B, d_x^B, u_x^B, a_x^B \rangle$. Let $\omega_x^{A:B} = \langle b_x^{A:B}, d_x^{A:B}, u_x^{A:B}, a_x^{A:B} \rangle$ be the opinion such that:*

$$\begin{cases} b_x^{A:B} = b_B^A \; b_x^B \\ d_x^{A:B} = b_B^A \; d_x^B \\ u_x^{A:B} = d_B^A + u_B^A + b_B^A \; u_x^B \\ a_x^{A:B} = a_x^B \end{cases}$$

then $\omega_x^{A:B}$ is called the uncertainty favouring discounted opinion of A. *By using the symbol $\otimes$ to designate this operation, we get $\omega_x^{A:B} = \omega_B^A \otimes \omega_x^B$.*

Definition 6 (Former Thm. 1 of [14]). *Let $\omega_x^A = \langle b_x^A, d_x^A, u_x^A, a_x^A \rangle$ and $\omega_x^B = \langle b_x^B, d_x^B, u_x^B, a_x^B \rangle$ be trust in x from A and B respectively. The opinion $\omega_x^{A\diamond B} = \langle b_x^{A\diamond B}, d_x^{A\diamond B}, u_x^{A\diamond B}, a_x^{A\diamond B} \rangle$ is then called the consensus between ω_x^A and ω_x^B, denoting the trust that an imaginary agent $[A, B]$ would have in x, as if that agent represented both A and B. In case of Bayesian (totally certain) opinions, their relative weight can be defined as $\gamma^{A/B} = \lim (u_x^B / u_x^A)$.*

Case I: $u_x^A + u_x^B - u_x^A \; u_x^B \neq 0$

$$\begin{cases} b_x^{A\diamond B} = \frac{b_x^A \; u_x^B + b_x^B \; u_x^A}{u_x^A + u_x^B - u_x^A \; u_x^B} \\ d_x^{A\diamond B} = \frac{d_x^A \; u_x^B + d_x^B \; u_x^A}{u_x^A + u_x^B - u_x^A \; u_x^B} \\ u_x^{A\diamond B} = \frac{u_x^A \; u_x^B}{u_x^A + u_x^B - u_x^A \; u_x^B} \\ a_x^{A\diamond B} = \frac{a_x^A \; u_x^B + a_x^B \; u_x^A - (a_x^A + a_x^B) \; u_x^A \; u_x^B}{u_x^A + u_x^B - 2 \; u_x^A \; u_x^B} \end{cases}$$

Case II: $u_x^A + u_x^B - u_x^A \; u_x^B = 0$

$$\begin{cases} b_x^{A\diamond B} = \frac{(\gamma^{A/B} \; b_x^A + b_x^B)}{(\gamma^{A/B} + 1)} \\ d_x^{A\diamond B} = \frac{(\gamma^{A/B} \; d_x^A + d_x^B)}{(\gamma^{A/B} + 1)} \\ u_x^{A\diamond B} = 0 \\ a_x^{A\diamond B} = \frac{(\gamma^{A/B} \; a_x^A + a_x^B)}{(\gamma^{A/B} + 1)} \end{cases}$$

By using the symbol '$\oplus$' to designate this operator, we can write $\omega_x^{A\diamond B} = \omega_x^A \oplus \omega_x^B$.

3 The Graphical Operators

In order to prove the soundness of our approach, we first need to discuss the geometry of Subjective Logic. For this purpose, let us recall, in Sect. 3.1 and 3.2, the main results from [6]. In Sect. 3.3 we recall the operator introduced in [6] and we show how to use it for discounting opinions. Section 3.4 is devoted to explain the operator for fusing opinions we introduce in this work. Please note that hereafter we will consider a fixed base rate of $\frac{1}{2}$.

3.1 The Geometry of Subjective Logic

A SL opinion $O \triangleq \langle b_O, d_O, u_O \rangle$ is a point in the $\mathbb{R}^3$ space, identified by the coordinate b_O for the first axis, d_O for the second axis, and u_O for the third axis. However, due to the requirement that $b_O + d_O + u_O = 1$, an opinion is a point inside (or at least on the edges of) the triangle $\overset{\triangle}{BDU}$ shown in Fig. 1, where $B = \langle 1, 0, 0 \rangle, D = \langle 0, 1, 0 \rangle, U = \langle 0, 0, 1 \rangle$.

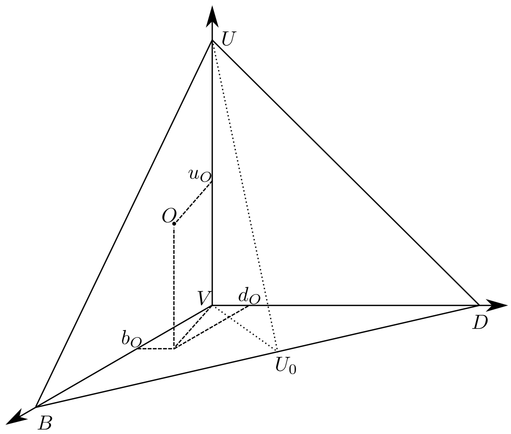

Fig. 1. The Subjective Logic plane region

Definition 7. *The* Subjective Logic plane region $\overset{\triangle}{BDU}$ *is the triangle whose vertices are the points* $B \triangleq \langle 1, 0, 0 \rangle$, $D \triangleq \langle 0, 1, 0 \rangle$, *and* $U \triangleq \langle 0, 0, 1 \rangle$ *on a* $\mathbb{R}^3$ *space where the axes are respectively the one of belief, disbelief, and uncertainty predicted by SL.*

Given this definition, we can easily define the *distance* between two opinions $\langle b_{O_1}, d_{O_1}, u_{O_1} \rangle$ and $\langle b_{O_2}, d_{O_2}, u_{O_2} \rangle$ as the Euclidean distance between the two point in the $\mathbb{R}^3$ space.

Definition 8. *Given two opinions* $O_1 = \langle b_{O_1}, d_{O_1}, u_{O_1} \rangle$ *and* $O_2 = \langle b_{O_2}, d_{O_2}, u_{O_2} \rangle$, *the* distance *between* O_1 *and* O_2 *is*

$$d(O_1, O_2) = \sqrt{(b_{O_2} - b_{O_1})^2 + (d_{O_2} - d_{O_1})^2 + (u_{O_2} - u_{O_1})^2}$$

Since an opinion is a point inside triangle $\overset{\triangle}{BDU}$, it can be mapped to a point in Fig. 2. This representation is similar to the one used in [5] for representing opinions in SL, but here the belief and disbelief axes are swapped.

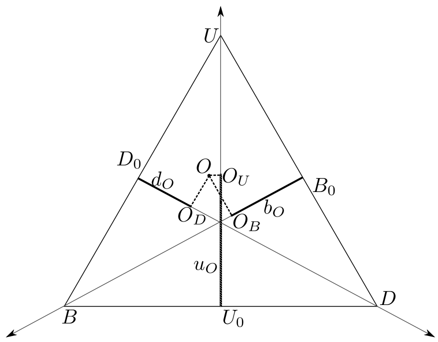

Fig. 2. An opinion $O \triangleq \langle b_O, d_O, u_O \rangle$ in SL after the $1 : \frac{\sqrt{3}}{\sqrt{2}}$ scale. The belief axis is the line from B_0 (its origin) toward the B vertex, the disbelief axis is the line from D_0 toward the D vertex, and the uncertainty axis is the line from U_0 toward the U vertex

In order to keep the discussion consistent with Jøsang's work [5], in what follows we will scale triangle $\overset{\triangle}{BDU}$ by a factor $1 : \frac{\sqrt{3}}{\sqrt{2}}$ thus obtaining that $|\overrightarrow{B_0B}| = |\overrightarrow{D_0D}| = |\overrightarrow{U_0U}| = 1$.

These geometric relations lie at the heart of the Cartesian transformation operator which is the subject of the next subsection.

3.2 The Cartesian Representation of Opinions

As shown in 3.1, an opinion in SL can be represented as a point in a planar figure (Fig. 2) laying on a Cartesian plane. In this section we will introduce the Cartesian transformation operator which returns the Cartesian coordinate of an opinion.

First of all, let us define the axes of the Cartesian system we will adopt.

Definition 9. *Given the SL plane region $\overset{\triangle}{BDU}$, the* associated Cartesian system *is composed by two axes, named respectively x, y, where the unit vector of the x axis $\overrightarrow{e_x} = \frac{1}{|\overrightarrow{BD}|}\overrightarrow{BD}$, the unit vector of the y axis $\overrightarrow{e_y} = \overrightarrow{e_u}$, and B is the origin.*

The correspondence between the three values of an opinion and the corresponding coordinate in the Cartesian system we defined is shown in the following proposition (proved in [6]).

Proposition 1 (Former Prop. 1 of [6]). *Given a SL plane region $\overset{\triangle}{BDU}$ and its associated Cartesian system $\langle x, y \rangle$, an opinion $O \triangleq \langle b_O, d_O, u_O \rangle$ is identified by the coordinate $\langle x_O, y_O \rangle$ s.t.:*

- $x_O \triangleq \dfrac{d_O + u_O\ \cos(\frac{\pi}{3})}{\sin(\frac{\pi}{3})}$
- $y_O \triangleq u_O$

3.3 The Graphical Discount Operator

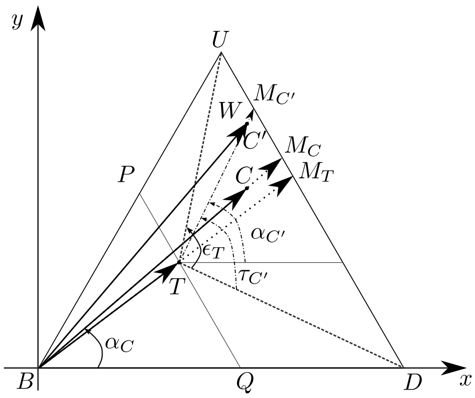

Fig. 3. Projection of the reputation opinion and combination with the trustworthiness opinion

In [6] we introduced an operator for combining a trustworthiness degree with a confidence degree. In this context we will use it for discounting opinions. Let us suppose that T is the opinion we have of the trustworthiness of an agent (i.e. the vector $\overrightarrow{BT}$ in Fig. 3), while C is the reputation that the agent has of an third agent (i.e. $\overrightarrow{BC}$ in Fig. 3). The operator defined in the following will return the opinion W such that $\overrightarrow{BW}$ is the sum of $\overrightarrow{BT}$ with $\overrightarrow{TC'}$, viz. the "projection" of $\overrightarrow{BC}$ in $\overset{\triangle}{UTD}$. In [6] we proved that the derived opinion benefits from interesting properties, for instance, b_W cannot be greater that b_T.

Definition 10. *Given the two opinions $T = \langle b_T, d_T, u_T\rangle$ and $C = \langle b_C, d_C, u_C\rangle$, the graphical-discount of C by T is $W = T \circ C$, where:*

- $u_W = u_T + \sin(\alpha_{C'})|\overrightarrow{TC'}|$
- $d_W = d_T + (u_T - u_W)\cos(\frac{\pi}{3}) + \cos(\alpha_{C'})\sin(\frac{\pi}{3})|\overrightarrow{TC'}|$

In particular:

- $\alpha_{C'} = \dfrac{\alpha_C\ \epsilon_T}{\frac{\pi}{3}} - \beta_T$
- $\alpha_C \triangleq \angle_{CBD} = \begin{cases} 0 & \textit{if } b_C = 1 \\ \arctan\left(\dfrac{u_C\ \sin(\frac{\pi}{3})}{d_C + u_C\ \cos(\frac{\pi}{3})}\right) & \textit{otherwise} \end{cases}$

- $\beta_T \triangleq \angle_{TDB} = \begin{cases} \dfrac{\pi}{3} & \textit{if } d_T = 1 \\ \arctan\left(\dfrac{u_T\ \sin(\frac{\pi}{3})}{1-(d_T + u_T\ \cos(\frac{\pi}{3}))}\right) & \textit{otherwise} \end{cases}$
- $\gamma_T \triangleq \angle_{TDU} = \dfrac{\pi}{3} - \beta_T$
- $\delta_T \triangleq \angle_{TUD} = \begin{cases} 0 & \textit{if } u_T = 1 \\ \arcsin\left(\dfrac{b_T}{|\overrightarrow{TU}|}\right) & \textit{otherwise} \end{cases}$
- $\epsilon_T \triangleq \angle_{DTU} = \pi - \gamma_T - \delta_T$
- $|\overrightarrow{TC'}| = \dfrac{|\overrightarrow{BC}|}{|\overrightarrow{BM_C}|}|\overrightarrow{TM_{C'}}| =$
  $= r_C\ |\overrightarrow{TM_{C'}}|$

with $r_C = \frac{|\overrightarrow{BC}|}{|\overrightarrow{BM_C}|}$, *and*

$$|\overrightarrow{TM_{C'}}| = \begin{cases} 2\ b_T & \textit{if } \alpha_{C'} = \dfrac{\pi}{2} \\ \dfrac{2}{\sqrt{3}}\ u_T & \textit{if } \alpha_{C'} = -\dfrac{\pi}{3} \\ \dfrac{2}{\sqrt{3}}\ (1-u_T) & \textit{if } \alpha_{C'} = \dfrac{2}{3}\pi \\ \dfrac{2\sqrt{\tan^2(\alpha_{C'})+1}}{|\tan(\alpha_{C'})+\sqrt{3}|}\ b_T & \textit{otherwise} \end{cases}$$

3.4 The Graphical Fusion Operator

Let us suppose we have n opinions $W_1, W_2, \ldots, W_n$ derived using the graphical operator $\circ$ s.t. $\forall i \in \{1, \ldots, n\}, W_i = T_i \circ C_i$. The fused opinion $\mathfrak{F}(W_1, W_2, \ldots, W_n)$ we want to obtain is the "balanced" centroid of the polygon determined by the n opinions.

We claim that a fusion operator must respect the following constraints:

R1: the fusion of an opinion $W_i = T_i \circ C_i$ must be balanced using $K_i = f(T_i)$ for some function $f(\cdot)$;
R2: if $\forall i, j\ K_i = K_j$, then the graphical fusion operator on $W_1, W_2, \ldots, W_n$, $\mathfrak{F}(W_1, W_2, \ldots, W_n)$ is the centroid of the polygon determined by n opinions;
R3: if $\exists i$ s.t. $K_i = 0$, then $\mathfrak{F}(W_1, \ldots, W_n) = \mathfrak{F}(W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_n)$.

Hereafter, we will consider $K_i = b_{T_i} + \frac{u_{T_i}}{2}$.

Definition 11. *Given the opinions* $T_1, T_2, \ldots, T_n, C_1, C_2, \ldots, C_n, W_1, W_2, \ldots, W_n$ *s.t.* $\forall i \in \{1 \ldots n\}, W_i = T_i \circ C_i$*, the opinion resulting from the fusion of opinions* $W_1, W_2, \ldots, W_n$ *is* $\langle b_{\mathfrak{F}(W_1,\ldots,W_n)}, d_{\mathfrak{F}(W_1,\ldots,W_n)}, u_{\mathfrak{F}(W_1,\ldots,W_n)}\rangle$ *where:*

- $b_{\mathfrak{F}(W_1,\ldots,W_n)} = \dfrac{1}{\sum_{i=1}^{n} K_i}\left(\sum_{i=1}^{n} K_i\ b_{W_i}\right)$
- $d_{\mathfrak{F}(W_1,\ldots,W_n)} = \dfrac{1}{\sum_{i=1}^{n} K_i}\left(\sum_{i=1}^{n} K_i\ d_{W_i}\right)$

– $u_{\mathfrak{F}(W_1,\ldots,W_n)} = \frac{1}{\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\, u_{W_i}\right)$

It is clear that this definition meets requirements $R1-3$. Moreover, we can also prove that $\mathfrak{F}(W_1,\ldots,W_n)$ is an opinion, and its Cartesian representation is the balanced centroid of the polygon identified by the points $W_1,\ldots,W_n$.

Proposition 2. *Given the opinions $T_1,T_2,\ldots,T_n,C_1,C_2,\ldots,C_n,W_1,W_2,\ldots,W_n$ s.t. $\forall i\in\{1\ldots n\}, W_i = T_i\circ C_i$, and $\langle b_{\mathfrak{F}(W_1,\ldots,W_n)}, d_{\mathfrak{F}(W_1,\ldots,W_n)}, u_{\mathfrak{F}(W_1,\ldots,W_n)}\rangle$ the opinion resulting from the fusion of opinions $W_1,W_2,\ldots,W_n$, then:*

i. $\langle b_{\mathfrak{F}(W_1,\ldots,W_n)}, d_{\mathfrak{F}(W_1,\ldots,W_n)}, u_{\mathfrak{F}(W_1,\ldots,W_n)}\rangle$ *is an opinion*

ii. $x_{\mathfrak{F}(W_1,\ldots,W_n)} = \frac{1}{\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\, x_{W_i}\right)$

iii. $y_{\mathfrak{F}(W_1,\ldots,W_n)} = \frac{1}{\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\, y_{W_i}\right)$

Proof. $(i.)$ *To prove that $\langle b_{\mathfrak{F}(W_1,\ldots,W_n)}, d_{\mathfrak{F}(W_1,\ldots,W_n)}, u_{\mathfrak{F}(W_1,\ldots,W_n)}\rangle$ is an opinion, we have to show that $u_{\mathfrak{F}(W_1,\ldots,W_n)} + d_{\mathfrak{F}(W_1,\ldots,W_n)} \leq 1$ holds.*

$$\begin{aligned}
u_{\mathfrak{F}(W_1,\ldots,W_n)} + d_{\mathfrak{F}(W_1,\ldots,W_n)} &= \frac{1}{\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i(u_{W_i}+d_{W_i})\right)\\
&= \frac{1}{\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i(1-b_{W_i})\right)\\
&= 1-\frac{1}{\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\, b_{W_i}\right)
\end{aligned}$$

$(ii.)$ *From Prop. 1,*

$$\begin{cases}
x_{\mathfrak{F}(W_1,\ldots,W_n)} = \frac{d_{\mathfrak{F}(W_1,\ldots,W_n)}}{\sin(\frac{\pi}{3})} + \frac{1}{2\ \sin(\frac{\pi}{3})\ \sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\, u_{W_i}\right)\\
x_{\mathfrak{F}(W_1,\ldots,W_n)} = \frac{1}{\sin(\frac{\pi}{3})\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\ (d_{W_i} + \frac{u_{W_i}}{2})\right)
\end{cases}$$

Thus we obtain:

$$\begin{aligned}
d_{\mathfrak{F}(W_1,\ldots,W_n)} = \sin(\frac{\pi}{3})&\left(\frac{1}{\sin(\frac{\pi}{3})\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\ (d_{W_i}+\frac{u_{W_i}}{2})\right)+\right.\\
&\left.-\frac{1}{2\ \sin(\frac{\pi}{3})\ \sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\, u_{W_i}\right)\right)\\
&= \frac{1}{\sum_{i=1}^n K_i}\left(\left(\sum_{i=1}^n K_i\ (d_{W_i}+\frac{u_{W_i}}{2})\right)-\left(\sum_{i=1}^n \frac{u_{W_i}}{2}\right)\right)\\
&= \frac{1}{\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\, d_{W_i}\right)
\end{aligned}$$

Since $\frac{1}{\sum_{i=1}^n K_i}\left(\sum_{i=1}^n K_i\, b_{W_i}\right) \geq 0$, *then* $u_{\mathfrak{F}(W_1,\ldots,W_n)} + d_{\mathfrak{F}(W_1,\ldots,W_n)} \leq 1$ *holds.*

$(iii.)$ *Immediate from Prop. 1.* □

4 Experimental Comparison

In order to determine the usefulness of our proposed operators, we evaluated them within the context of a trust framework. In this section, we describe our experimental setup and present the results.

4.1 The Experiment

In this experiment each agent can communicate with all the other agents in the network. In order to randomly generate these networks, we consider a variable $P^L \in [0, 1]$ representing the probability that an agent is connected[1] to another agent (we exclude the self-connections). Note that we do not constrain connections to be bidirectional[2]. In this experiment we consider P^L varying between 5 and 25 with a step of 5. For each of the value of P^L, we execute the following steps.

Trust System Construction. We build a set of 50 agents $\mathcal{A} = \{a_1, \ldots, a_{50}\}$: each agent a_x is characterised by a knowledge base $\mathcal{KB}_{a_x}$ and by the probability of responding truthfully to another agent's query, namely $P^T_{a_x} \in [0 \ldots 1]$. For each agent a_x, we randomly choose $P^T_{a_x}$.

We also require that $(\Omega = \top) \in \mathcal{KB}_{a_x}$. In other terms, all the agents share the same information $(\Omega = \top)$ to be read "a_x knows that Ω is $\top$".

For each agent a_x, we determine if it can communicate with $a_y \neq a_x$ according to P^L: if a_y is connected to a_x, then we say that a_y is a connection of a_x ($a_y \in N_{a_x}$).

Then, we enter in the core of the experiment, which considers two phases: (1) the bootstrapping, and (2) the exploration.

During the bootstrapping phase, each agent queries each agent with which it is connected in order to determine its degrees of trustworthiness.

After the bootstrapping phase, an agent "explorer" determines the trustworthiness degree of each agent in the network with the specific aim of minimising the number of messages exchanged.

Trust System Bootstrapping. The bootstrapping phase follows the intuition behind [16], where a β distribution is used for analysing repetitive experiments and deriving a SL opinion. In this experiment, each agent a_x asks each agent $a_y \in N_{a_x}$ about the shared information Ω a number of times equals to $\#_B$. Each time, a_y answers either truthfully or not according to $P^T_{a_y}$ only (the communications are stateless). Therefore, the two possible answers of a_y are: $\Omega = \top$ or $\Omega = \perp$ (which is the case where a_y lies).

[1] The term "connection" here can be declined in different contexts, like "friend" in Facebook, or "follower" in Twitter.

[2] Although this may seem counter-intuitive, it partially captures real-world social media: for instance Twitter messages are public, therefore we don't know who will read our messages. The same applies with slightly modifications to Google Plus too, and, of course, to blogging activities in general.

Agent a_x counts the number of exchanges when agent a_y answered truthfully ($\#_\top$) and when it lied ($\#_\bot$). Clearly, $\#_B = \#_\top + \#_\bot$. Then, according to [16], a_x has an opinion on a_y's degree of trustworthiness

$$O_{a_x}^{a_y} = \langle \frac{\#_\top}{\#_B + 2}, \frac{\#_\bot}{\#_B + 2}, \frac{2}{\#_B + 2} \rangle$$

which should be close (according to the definition of distance given in Def. 8) to the "ideal" ("real") opinion the (omniscient) experimenter has on a_y, viz.

$$O_{Exp}^{a_y} = \langle P_{a_y}^T, (1 - P_{a_y}^T), 0 \rangle$$

Therefore, for each agent $a_y \in N_{a_x}$, a_x saves its opinion on a_y in its knowledge base (i.e. $\forall a_y \in a_x, O_{a_x}^{a_y} \in \mathcal{KB}_{a_x}$).

We collected data for ten different values of $\#_B$ varying it between 25 and 250 with a step of 25. Since this parameter does not influence the results, we silently drop the detail concerning $\#_B$.

Exploring the Network. After each agent has enriched its knowledge base with the opinions on its connections' trustworthiness, an "explorer" $a_S \in \mathcal{A}$ is randomly selected and it is asked to determine the trustworthiness of each agent in the network. a_S acquires information asking its connections "Which are your connections?". Each agent $a_y \in N_{a_S}$ answers this question according to $P_{a_y}^T$: therefore the answer to that question is $NewAgents_{a_y} \subseteq \mathcal{A}$ such that $NewAgents_{a_y} \subseteq N_{a_y}$ (clearly if $P_{a_y}^T = 1$, then $NewAgents_{a_y} = N_{a_y}$).

Agent a_S collects all the answers and creates the following set of pairs:

$$\mathcal{M} = \{\langle a_z, \{a_{y_1}, \ldots, a_{y_n}\}\rangle \mid \forall i \in [1 \ldots n]\ a_{y_i} \in N_{a_S} \text{ and } a_z \in \bigcap_{i=1}^{n} N_{a_{y_i}}\}$$

Then, for each pair of $\mathcal{M}$, $\langle a_z, \{a_{y_1}, \ldots, a_{y_n}\}\rangle$, such that $a_z \notin N_{a_S} \cup \{a_S\}$, a_S asks each a_{y_i} about $O_{a_{y_i}}^{a_z}$. a_{y_i} answers according to $P_{a_{y_i}}^T$ either $O_{a_{y_i}}^{a_z}$ or $\langle b_R, d_R, u_R\rangle$ where $\langle b_R, d_R, u_R\rangle$ is a subjective logic opinion computed randomly such that $\langle b_R, d_R, u_R\rangle \neq O_{a_{y_i}}^{a_z}$. Since a_S cannot determine whether the answer is true or not, hereafter we abuse notation, identifying with $O_{a_{y_i}}^{a_z}$ the answer a_S received from a_{y_i} at the question "What is your opinion about a_z".

Subsequently, a_S computes $O_{a_S|J}^{a_z} = (O_{a_S}^{a_{y1}} \otimes O_{a_{y_1}}^{a_z}) \oplus \ldots \oplus (O_{a_S}^{a_{yn}} \otimes O_{a_{y_n}}^{a_z})$ (viz. the fusion of the discounted opinions on a_z of its connections using Jøsang operators), and $O_{a_S|G}^{a_z} = \mathfrak{F}((O_{a_S}^{a_{y1}} \circ O_{a_{y_1}}^{a_z}), \ldots, (O_{a_S}^{a_{yn}} \circ O_{a_{y_n}}^{a_z})$ (viz. the fusion of the discounted opinions on a_z of its connections using the graphical operators introduced in Def. 10 and 11).

Finally, each agent a_z is added to the list of the connections of a_S and the process starts again by setting $\mathcal{M} = \emptyset$ and querying each member of the connections until, in two subsequent interactions, no further agents are added to a_S's connections.

Computing the Distances. For each agent $a_z \in \mathcal{A} \setminus \{a_S\}$ we compute the distance between the two derived opinions $O^{a_z}_{a_S|J}$ and $O^{a_z}_{a_S|G}$, and the "ideal" opinion $O^{a_z}_{Exp}$. $\forall a_z \in \mathcal{A} \setminus \{a_S\}$, $d(O^{a_z}_{a_S|J}, O^{a_z}_{Exp})$ is the distance between the derived opinion using Jøsang's operators and the "ideal" one (abbrev. d_J), and $d(O^{a_z}_{a_S|G}, O^{a_z}_{Exp})$ is the distance between the derived opinion using the graphical operators and the "ideal" one (abbrev. d_G).

Finally, for each $a_z \in \mathcal{A} \setminus \{a_S\}$ we compare the two computed distances obtaining the following scalar comparison value:

$$r(a_z) = \begin{cases} -\log \dfrac{d(O^{a_z}_{a_S|G}, O^{a_z}_{Exp})}{d(O^{a_z}_{a_S|J}, O^{a_z}_{Exp})} & \text{if } d(O^{a_z}_{a_S|G}, O^{a_z}_{Exp}) > d(O^{a_z}_{a_S|J}, O^{a_z}_{Exp}) \\ \log \dfrac{d(O^{a_z}_{a_S|J}, O^{a_z}_{Exp})}{d(O^{a_z}_{a_S|G}, O^{a_z}_{Exp})} & \text{if } d(O^{a_z}_{a_S|J}, O^{a_z}_{Exp}) \geq d(O^{a_z}_{a_S|G}, O^{a_z}_{Exp}) \end{cases}$$

Since we are interested in the average case, a_S explores the network $|\mathcal{A}|/2 = 25$ times, and with a little abuse of notation we identify with $r(a_z)$ the average of the 25 computed logarithmic ratios. Moreover $\overline{r(a_z)} = average_{a_z \in \mathcal{A} \setminus \{a_S\}} r(a_z)$ is the average of the comparison value over the whole set of agents.

4.2 Results

Due to space constraints, we consider only the values of $\overline{r(a_z)}$ in function of the percentage of connections P^L. To ensure that the outcomes are not biased by the random generator, we run the same experiment ten times. Each run follows the steps described in Sect. 4.1, and thus for each value of P^L, 10 networks have been generated randomly. Moreover, since each agent can lie, each generated network has been explored 25 times. Therefore, for each run, for each value of P^L, 250 explorations over 10 different networks have been carried on (overall, 12500 explorations have been considered in this experiment).

Figure 4 shows the results of this experiment, where for each run, for each value of P^L, the averaged (over the 250 explorations) value of $\overline{r(a_z)}$ is depicted. Let us recall here that if, in Fig. 4, the depicted value is 0, then on the average the two operators return opinions that are at the same distance from the "ideal" one. Positive values indicate that the graphical operators returned (on the average) values closer to the "ideal" one than Jøsang's and the absolute value shows the average logarithmic ratio between the two distances. Negative values, on the other hand, demonstrate that Jøsang's operators return opinions closer to the "ideal" ones than the graphical operators.

Figure 4 suggests that there is a strict relationship between P^L and the average of the comparison. In fact for small values of P^L, the graphical operators return trust opinions much closer to the "ideal" ones that Jøsang's ones. Moreover, for greater values of P^L, the results indicate that the greater P^L, the more the results of the graphical operators are "close" to the Jøsang operators (i.e. the more distances of the opinions computed by the two sets of operators from the "ideal" ones are close). This probably depends on the

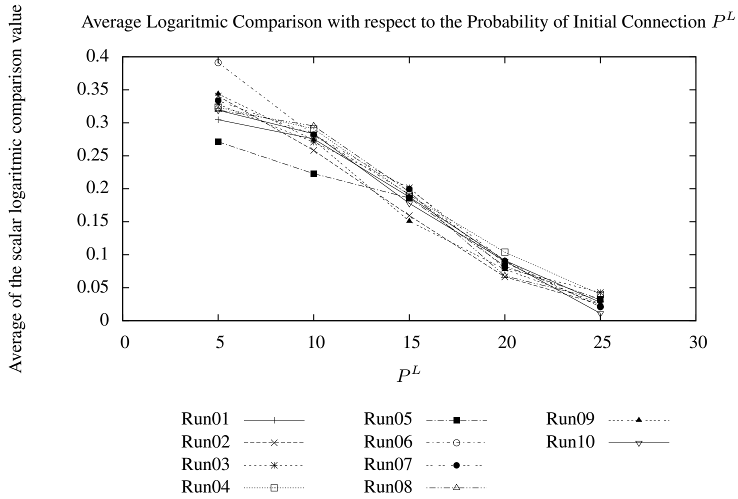

Fig. 4. Average Logaritmic Comparison with respect to the Probability of Initial Connections P^L: the 10 runs of the experiment.

fact that the more the graph becomes dense, the more are the trustworthiness opinions computed during the bootstrap, the less are the trustworthiness opinions derived using the operators.

We conclude our empirical evaluation with Fig. 5, which depicts the ratio (averaged over the ten runs) between the two operators on a linear scale (whereas a logarithmic scale was used until now). From this figure we can observe that:

1. for small values of P^L, the graphical operators return opinions closer to the "ideal" one than Jøsang's of a factor 2 (on the average $\frac{d_J}{d_G} \simeq 2$), or, in other terms, approx. 50% closer;
2. the greater the P^L, the more similar the performance of the two sets of operators, the smaller the standard deviation on the results obtained by the experiments;
3. the overall average on the linear scale is 1.56.

In conclusion, this experiment shows that the smaller the number of communication links, the greater the performance of the graphical operators, which, on the average, returns opinions that are 36% (ratio 1.56) closer to the "ideal" ones than Jøsang operators.

5 Conclusions and Future Works

The discount and the fusion operators play an important role in standard Subjective Logic, and form the core of the Beta Reputation System. In fact, they are used to combine and discount reputation information from multiple agents within a trust network.

Following our earlier work in [6], where we described a graphical operator for discounting opinions, this paper introduces a graphical fusion operator. Given the pivotal

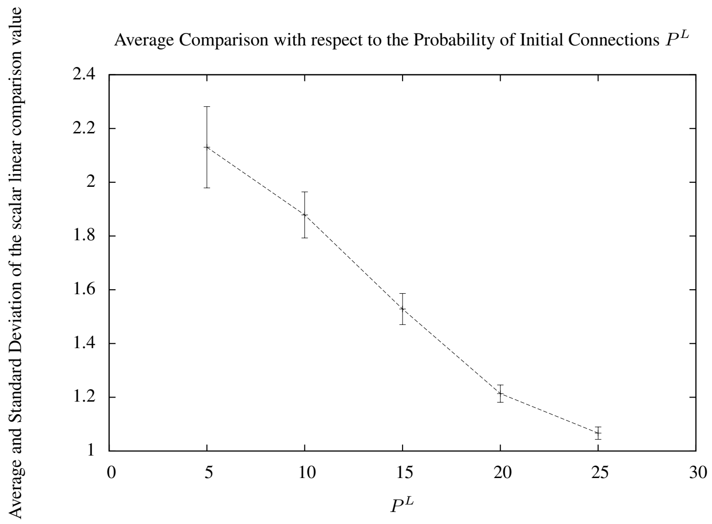

Fig. 5. Average and Standard Deviation of the linear comparison value with respect to the Probability of Initial Connections P^L: average and standard deviation over the 10 runs of the experiment.

role these operators play in trust mechanisms, we evaluated the proposed graphical operators in a trust scenario, substituting the new operators for the standard ones.

The results shown in Sect. 4.2 suggests that when we have to compute trustworthiness opinion of agents of which we have just reputation opinions from other agents, the graphical operators determines opinions that are up-to the 50% (36% on average) closer to the ground truth than Jøsang's operators. More important, they illustrate that when the considered trust system that can be represented as a sparse graph, which is common in real cases cf. [17], then (on the average) the graphical operators outperform Jøsang's ones.

An empirical evaluation of the graphical operators on real trust systems is already envisaged as the main future work. In addition, we want to develop, where possible, graphical operators analogous to other Subjective Logic operators, and we intend to study these, as well as investigate their properties.

Acknowledgements The authors thank the anonymous reviewers for their helpful comments.

Research was sponsored by US Army Research laboratory and the UK Ministry of Defence and was accomplished under Agreement Number W911NF-06-3-0001. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the US Army Research Laboratory, the U.S. Government, the UK Ministry of Defense, or the UK Government. The US and UK Governments are authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation hereon.

References

1. Sabater, J., Sierra, C.: Review on computational trust and reputation models. Artificial Intelligence Review **24** (2005) 33–60
2. Haghpanah, Y., des Jardins, M.: Prep: a probabilistic reputation model for biased societies. In: Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems - Volume 1. AAMAS '12 (2012) 315–322
3. Jøsang, A., Azderska, T., Marsh, S.: Trust transitivity and conditional belief reasoning. In Dimitrakos, T., Moona, R., Patel, D., McKnight, D.H., eds.: IFIPTM. Volume 374 of IFIP Advances in Information and Communication Technology., Springer (2012) 68–83
4. Burnett, C., Oren, N.: Sub-delegation and trust. In: Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems - Volume 3. AAMAS '12 (2012) 1359–1360
5. Jøsang, A.: A logic for uncertain probabilities. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **9**(3) (2001) 279–311
6. Cerutti, F., Toniolo, A., Oren, N., Norman, T.J.: Context-dependent trust decisions with subjective logic. Technical Report ABDN–CS–2013–05–01, Department of Computing Science, University of Aberdeen (May 2013)
7. Castelfranchi, C., Falcone, R.: Trust theory: A socio-cognitive and computational model. Wiley Series in Agent Technology (2010)
8. Urbano, J., Rocha, A., Oliveira, E.: A socio-cognitive perspective of trust. In Ossowski, S., ed.: Agreement Technologies. Volume 8 of Law, Governance and Technology Series. Springer Netherlands (2013) 419–429
9. Jøsang, A., Ismail, R., Boyd, C.: A survey of trust and reputation systems for online service provision. Decision Support Systems **43**(2) (2007) 618–644
10. Dempster, A.P.: A generalization of bayesian inference. Journal of the Royal Statistical Society. Series B (Methodological) **30**(2) (1968) pp. 205–247
11. Shafer, G.: A Mathematical Theory of Evidence. Princeton University Press (1976)
12. Jøsang, A., McAnally, D.: Multiplication and comultiplication of beliefs. International Journal of Approximate Reasoning **38**(1) (2004) 19–51
13. Jøsang, A., Pope, S., Daniel, M.: Conditional deduction under uncertainty. In: Proceedings of the 8th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2005). (2005)
14. Jøsang, A., Pope, S., Marsh, S.: Exploring different types of trust propagation. In: Proceedings of the 4th International Conference on Trust Management (iTrust'06). (2006)
15. McAnally, D., Jøsang, A.: Addition and subtraction of beliefs. In: Proceedings of the conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU2004). (2004)
16. Ismail, R., Jøsang, A.: The beta reputation system. In: Prooceedings of BLED 2002. (2002)
17. Guha, R., Kumar, R., Raghavan, P., Tomkins, A.: Propagation of trust and distrust. In: Proceedings of the 13th conference on World Wide Web - WWW '04, New York, New York, USA, ACM Press (May 2004) 403