A General QBF-based Formalization of Abstract Argumentation Theory

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Abstract. We introduce a unified logical approach, based on signed theories and Quantified Boolean Formulas (QBFs), that can serve as a basis for representing and reasoning with various argumentation-based decision problems. By this, we are able to represent, in a uniform and simple way, a wide range of extension-based semantics for argumentation theory, including complete, grounded, preferred, semi-stable, stage, ideal and eager semantics. Furthermore, our approach involves only propositional languages and quantifications over propositional variables, making decision problems like skeptical and credulous acceptance of arguments simply a matter of logical entailment and satisfiability, which can be verified by existing QBF-solvers.

Keywords. argumentation semantics, quantified Boolean formulas, 4-valued logics

1. Introduction

Dung's approach to argumentation theory [13] has led to a wide range of argumentation semantics being proposed in the literature, including grounded, complete, preferred and stable semantics [13], semi-stable semantics [9,20], stage semantics [20], ideal semantics [14] and eager semantics [10]. One particular issue that has been studied is how these semantics can be expressed using *purely logical* formalizations. Complete semantics, for instance, can be expressed in propositional logic [12], and grounded, preferred, stable and semi-stable semantics can be expressed using second-order modal logic [17,18].

In this paper we show that a wide range of known Dung-style semantics, including *all* of the above mentioned argumentation semantics, can be adequately represented in a uniform and simple way that is based on propositional languages and quantifications over propositional variables. For this, we incorporate signed theories and quantified Boolean formulas (QBFs). We first show how complete semantics can be described using four-valued logics and signed theories. Based on this result, we then continue to model grounded, preferred, stable, semi-stable, ideal and eager semantics, using an approach based on quantified Boolean formulas, similar to the one taken in [1,3] for reasoning with paraconsistent preferential entailments, and in [4] for repairing inconsistent databases. To illustrate that our approach is not restricted to admissibility-based semantics, we show how the notion of stage semantics can also be represented in our framework.

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2. Semantics for Argumentation Frameworks

First, we briefly review some basic definitions of argumentation theory, based on Dung's seminal work [13].

Definition 1 A (finite) *argumentation framework* (AF) is a pair $\mathscr{A} = \langle Ar, att \rangle$, where Ar is a finite set, the elements of which are called *arguments*, and *att* is a binary relation on $Ar \times Ar$ whose instances are called *attacks*. When $(A,B) \in att$ we say that A *attacks* B (or that B is *attacked by* A).

Let $\mathscr{A} = \langle Ar, att \rangle$ be an AF, $A \in Ar$, and $Args \subseteq Ar$. We denote by A^+ the arguments attacked by A, i.e., $A^+ = \{B \in Ar \mid att(A,B)\}$, and by A^- the arguments that attack A, i.e., $A^- = \{B \in Ar \mid att(B,A)\}$. The set of arguments that are attacked by some argument in $Args$ and the arguments that attack some argument in $Args$ are, respectively, $Args^+ = \cup_{A \in Args} A^+$ and $Args^- = \cup_{A \in Args} A^-$. The set $Args \cup Args^+$ is called the *range* of $Args$.

One of the key questions of argumentation theory is what are the combinations of arguments that can collectively be accepted for a given argumentation framework. Next, we recall the two standard approaches for answering this question.

2.1. Extension-Based Semantics

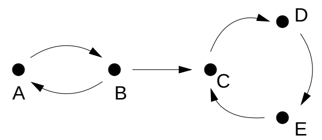
The extension-based approach defines sets of arguments (called *extensions*) that can collectively be accepted in a framework. For defining different kinds of extensions for an argumentation framework, the following notions are used.

Definition 2 Let $\mathscr{A} = \langle Ar, att \rangle$ be an AF, $A \in Ar$ an argument, and $Args \subseteq Ar$ a set of arguments. $Args$ is *conflict-free* iff $Args \cap Args^+ = \emptyset$, $Args$ *defends* A iff $A^- \subseteq Args^+$, and $F(Args) = \{A \in Ar \mid A^- \subseteq Args^+\}$.

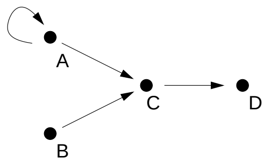
A set $Args$ is therefore conflict-free if there is no attack between its arguments, an argument A is defended by $Args$ if any argument that attacks A is attacked by $Args$, and $F(Args)$ is the set of arguments that are defended by $Args$.

Definition 3 Let $\mathscr{A} = \langle Ar, att \rangle$ be an AF. A conflict-free set $Args \subseteq Ar$ is called

- an *admissible set* of $\mathscr{A}$, iff $Args \subseteq F(Args)$,
- a *complete extension* of $\mathscr{A}$, iff $Args = F(Args)$,
- the *grounded extension* of $\mathscr{A}$, iff it is the minimal complete extension of $\mathscr{A}$,[2]
- a *preferred extension* of $\mathscr{A}$, iff it is a maximal complete extension of $\mathscr{A}$,
- the *ideal extension* of $\mathscr{A}$ iff it is the maximal complete extension that is a subset of each preferred extension of $\mathscr{A}$,
- a *stable extension* of $\mathscr{A}$, iff it is a complete extension of $\mathscr{A}$ and $Args^+ = Ar \setminus Args$,
- a *semi-stable extension* of $\mathscr{A}$, iff it is a complete extension of $\mathscr{A}$ with maximal range among all complete extensions of $\mathscr{A}$,
- the *eager extension* of $\mathscr{A}$, iff it is the maximal complete extension that is a subset of each semi-stable extension of $\mathscr{A}$,
- a *stage extension* of $\mathscr{A}$ iff it has a maximal range among all conflict-free sets of $\mathscr{A}$.



The argumentation framework $\mathscr{A}_1$



The argumentation framework $\mathscr{A}_2$

Example 4 Consider the argumentation frameworks $\mathscr{A}_1$ and $\mathscr{A}_2$. The admissible sets of $\mathscr{A}_1$ are $\emptyset$, $\{A\}$, $\{B\}$ and $\{B,D\}$, its complete extensions are $\emptyset$, $\{A\}$, and $\{B,D\}$, the grounded extension is $\emptyset$, the preferred extensions are $\{A\}$ and $\{B,D\}$, the ideal extension is $\emptyset$, the stable extension is $\{B,D\}$, and this is also the only semi-stable extension, eager extension, and stage extension of $\mathscr{A}_1$.

As for $\mathscr{A}_2$, its conflict-free sets are $\emptyset$, $\{B\}$, $\{C\}$, $\{D\}$ and $\{B,D\}$, and the admissible sets are $\emptyset$, $\{B\}$ and $\{B,D\}$. This time there is just one complete extension, $\{B,D\}$, which is also the only grounded, preferred, ideal, semi-stable, eager and stage extension of $\mathscr{A}_2$. Note that $\mathscr{A}_2$ does not have any stable extension.

2.2. Labeling-Based Semantics

Argument labelings [8,12] provide an alternative way to describe argumentation semantics.

Definition 5 Let $\mathscr{A} = \langle Ar, att\rangle$ be an AF. An *argument labeling* is a complete function $lab : Ar \to \{\mathsf{in}, \mathsf{out}, \mathsf{undec}\}$. We shall sometimes write $\mathsf{In}(lab)$ for $\{A \in Ar \mid lab(A) = \mathsf{in}\}$, $\mathsf{Out}(lab)$ for $\{A \in Ar \mid lab(A) = \mathsf{out}\}$ and $\mathsf{Undec}(lab)$ for $\{A \in Ar \mid lab(A) = \mathsf{undec}\}$.

In essence, a labeling expresses which arguments are accepted (labeled in), rejected (labeled out) and have a neutral status (labeled undec). Since a labeling *lab* is a partition of *Ar*, we sometimes write it as a triple $\langle \mathsf{In}(lab), \mathsf{Out}(lab), \mathsf{Undec}(lab)\rangle$.

Definition 6 Consider the following conditions:

(Pos1)	If $lab(A) = \mathsf{in}$ then there is no $B \in A^-$ such that $lab(B) = \mathsf{in}$.
(Pos2)	If $lab(A) = \mathsf{in}$ then $lab(B) = \mathsf{out}$ for all $B \in A^-$.
(Neg)	If $lab(A) = \mathsf{out}$ then there is $B \in A^-$ such that $lab(B) = \mathsf{in}$.
(Neither)	If $lab(A) = \mathsf{undec}$ then there is $B \in A^-$ such that $lab(B) \neq \mathsf{out}$ and there is no $B \in A^-$ such that $lab(B) = \mathsf{in}$.

A labeling *lab* for $\mathscr{A} = \langle Ar, att\rangle$ is called *conflict-free* iff (for every $A \in Ar$) it satisfies conditions **(Pos1)** and **(Neg)**, *admissible* if it satisfies conditions **(Pos2)** and **(Neg)**, and *complete* iff it is admissible and also satisfies **(Neither)**.

Definition 7 Let $Comp(\mathscr{A})$ denote the complete labelings of $\mathscr{A} = \langle Ar, att\rangle$. A labeling $lab_c \in Comp(\mathscr{A})$ of is called

- the *grounded labeling* of $\mathscr{A}$, iff $\mathsf{In}(lab_c) \in \min\{\mathsf{In}(lab) \mid lab \in Comp(\mathscr{A})\}$,[3]
- a *preferred labeling* of $\mathscr{A}$, iff $\mathsf{In}(lab_c) \in \max\{\mathsf{In}(lab) \mid lab \in Comp(\mathscr{A})\}$,
- a *stable labeling* of $\mathscr{A}$, iff $\mathsf{Undec}(lab_c) = \emptyset$.
- a *semi-stable labeling* of $\mathscr{A}$ iff $\mathsf{Undec}(lab_c) \in \min\{\mathsf{Undec}(lab) \mid lab \in Comp(\mathscr{A})\}$,

[2] Here and elsewhere in this definition the minimum and maximum are taken with respect to set inclusion.
[3] We assume that min (resp. max) selects those sets that are minimal (resp. maximal) w.r.t. set inclusion.

– the *ideal labeling* of $\mathscr{A}$, iff $\mathsf{In}(lab_c) \in \max\{\mathsf{In}(lab) \mid lab \in Comp(\mathscr{A})$ and also $\mathsf{In}(lab) \subseteq \mathsf{In}(lab')$ for every preferred labeling lab' of $\mathscr{A}\}$,
– the *eager labeling* of $\mathscr{A}$, iff $\mathsf{In}(lab_c) \in \max\{\mathsf{In}(lab) \mid lab \in Comp(\mathscr{A})$ and also $\mathsf{In}(lab) \subseteq \mathsf{In}(lab')$ for every semi-stable labeling lab' of $\mathscr{A}\}$,
– a *stage labeling* of $\mathscr{A}$ is a conflict-free labeling lab_{cf} of $\mathscr{A}$ so that $\mathsf{Undec}(lab_{cf}) \in \min\{\mathsf{Undec}(lab) \mid lab$ is a conflict-free labeling of $\mathscr{A}\}$.

A one-to-one correspondence between the grounded (respectively: preferred, stable, semi-stable) labelings of $\mathscr{A}$ and its grounded (respectively: preferred, stable, semi-stable) extensions is shown in [12]. A similar correspondence between the ideal (respectively stage) labelings of $\mathscr{A}$ and its ideal (respectively stage) extensions is shown in [11].

3. Semantics for Signed QBF-Based Theories

As indicated previously, our purpose is to provide a third, logic-based, perspective on argumentation frameworks. Below, we define the framework for doing so (see also [1]).

3.1. Four-Valued Semantics and Signed Formulas

Consider the truth values t ('true'), f ('false'), $\bot$ ('neither true nor false') and $\top$ ('both true and false'). These elements may be arranged in a lattice structure in which f is the minimal element, t is the maximal one, and the other two values are intermediate elements that are incomparable. The corresponding lattice $\mathscr{FOUR} = (\{t, f, \top, \bot\}, \leq)$ intuitively reflects differences in the 'measure of truth' of its elements. This is a distributive lattice with an order reversing involution $\neg$, for which $\neg t = f$, $\neg f = t$, $\neg\top = \top$ and $\neg\bot = \bot$. We shall denote the meet and the join of this lattice by $\wedge$ and $\vee$, respectively. Another useful operator is: $a \supset b = t$ if $a \in \{f, \bot\}$, and $a \supset b = b$ otherwise.[4]

The four truth values may be represented by pairs of two-valued components of the lattice $(\{0,1\}, 0 < 1)$ by: $t = (1,0)$, $f = (0,1)$, $\top = (1,1)$, $\bot = (0,0)$. Intuitively, if a formula ψ has the value (x,y), then x indicates whether ψ should be accepted and y indicates whether ψ should be rejected. The basic operators of $\mathscr{FOUR}$ may be expressed in terms of this representation as follows:

Lemma 8 *Let* $x_1, x_2, y_1, y_2 \in \{0,1\}$. *Then:* $(x_1,y_1) \vee (x_2,y_2) = (x_1 \vee x_2, y_1 \wedge y_2)$, $(x_1,y_1) \wedge (x_2,y_2) = (x_1 \wedge x_2, y_1 \vee y_2)$, $(x_1,y_1) \supset (x_2,y_2) = (\neg x_1 \vee x_2, x_1 \wedge y_2)$, *and* $\neg(x,y) = (y,x)$.

In our context, the four values above are used for evaluating formulas in a propositional language $\mathscr{L}$, consisting of a (countably infinite) set of atomic formulas $\mathsf{Atoms}(\mathscr{L})$, the propositional constants t and f, and the connectives $\neg, \wedge, \vee, \supset$. As usual, a *valuation* ν for $\mathscr{L}$ is a function from $\mathsf{Atoms}(\mathscr{L})$ to $\{t, f, \bot, \top\}$ so that $\nu(\mathsf{t}) = t$ and $\nu(\mathsf{f}) = f$. Any valuation is extended to complex formulas in the usual way. A valuation ν *satisfies* ψ iff $\nu(\psi) \in \{t, \top\}$. A valuation that satisfies every formula in a set of formulas (theory) $\mathscr{T}$ is a *model* of $\mathscr{T}$. The set of models of $\mathscr{T}$ is denoted by $mod(\mathscr{T})$. Now, it is obvious that the representation of truth values in terms of pairs of two-valued components implies a similar way of representing four-valued valuations. A four-valued

[4]See, e.g., [2,5] for further discussions on $\mathscr{FOUR}$ and the logics that are induced by this structure.

valuation ν may be represented in terms of a pair of two-valued components (ν_1, ν_2) by $\nu(p) = (\nu_1(p), \nu_2(p))$. So if, for instance, $\nu(p) = t$, then $\nu_1(p) = 1$ and $\nu_2(p) = 0$. Note also that $\nu = (\nu_1, \nu_2)$ is a four-valued model of $\mathscr{T}$ iff $\nu_1(\psi) = 1$ for every $\psi \in \mathscr{T}$.

Definition 9 A *signed alphabet* $\mathsf{Atoms}^{\pm}(\mathscr{L})$ is a set consisting of two symbols $p^{\oplus}, p^{\ominus}$ for each atom $p \in \mathsf{Atoms}(\mathscr{L})$. The language over $\mathsf{Atoms}^{\pm}(\mathscr{L})$ is denoted by $\mathscr{L}^{\pm}$.

- The two-valued valuation ν^2 on $\mathsf{Atoms}^{\pm}(\mathscr{L})$ that is *induced by* a four-valued valuation $\nu^4 = (\nu_1, \nu_2)$ on $\mathsf{Atoms}(\mathscr{L})$, interprets $p^{\oplus}$ as $\nu_1(p)$ and $p^{\ominus}$ as $\nu_2(p)$.
- The four-valued valuation ν^4 on $\mathsf{Atoms}(\mathscr{L})$ that is *induced by* a two-valued valuation ν^2 on $\mathsf{Atoms}^{\pm}(\mathscr{L})$ is defined by $\nu^4(p) = (\nu^2(p^{\oplus}), \nu^2(p^{\ominus}))$.

Definition 10 For an atom p and formulas ψ, ϕ, define the following formulas in $\mathscr{L}^{\pm}$:

$$
\begin{array}{ll}
\tau_1(p) = p^{\oplus}, & \tau_2(p) = p^{\ominus}, \\
\tau_1(\neg\psi) = \tau_2(\psi), & \tau_2(\neg\psi) = \tau_1(\psi), \\
\tau_1(\psi \wedge \phi) = \tau_1(\psi) \wedge \tau_1(\phi), & \tau_2(\psi \wedge \phi) = \tau_2(\psi) \vee \tau_2(\phi), \\
\tau_1(\psi \vee \phi) = \tau_1(\psi) \vee \tau_1(\phi), & \tau_2(\psi \vee \phi) = \tau_2(\psi) \wedge \tau_2(\phi), \\
\tau_1(\psi \supset \phi) = \neg\tau_1(\psi) \vee \tau_1(\phi), & \tau_2(\psi \supset \phi) = \tau_1(\psi) \wedge \tau_2(\phi).
\end{array}
$$

We denote $\tau_i(\mathscr{T}) = \{\tau_i(\psi) \mid \psi \in \mathscr{T}\}$ $(i = 1, 2)$

Example 11 $\tau_1(\neg(p \wedge \neg q) \supset \neg q) = \neg\tau_1(\neg(p \wedge \neg q)) \vee \tau_1(\neg q) = \neg\tau_2(p \wedge \neg q) \vee \tau_2(q) = \neg(\tau_2(p) \vee \tau_2(\neg q)) \vee \tau_2(q) = \neg(\tau_2(p) \vee \tau_1(q)) \vee \tau_2(q) = \neg(p^{\ominus} \vee q^{\oplus}) \vee q^{\ominus}$.

Proposition 12 [1] *If ν^2 is induced by ν^4 or ν^4 is induced by ν^2, then ν^4 satisfies ψ iff ν^2 satisfies $\tau_1(\psi)$, and ν^4 satisfies $\neg\psi$ iff ν^2 satisfies $\tau_2(\psi)$.*

Definition 13 For a formula ψ in $\mathscr{L}$ we define the following signed formulas in $\mathscr{L}^{\pm}$:

$$
\begin{array}{ll}
\mathsf{val}(\psi, t) = \tau_1(\psi) \wedge \neg\tau_2(\psi), & \mathsf{val}(\psi, f) = \neg\tau_1(\psi) \wedge \tau_2(\psi), \\
\mathsf{val}(\psi, \top) = \tau_1(\psi) \wedge \tau_2(\psi), & \mathsf{val}(\psi, \bot) = \neg\tau_1(\psi) \wedge \neg\tau_2(\psi).
\end{array}
$$

Proposition 14 *If ν^2 is induced by ν^4, or ν^4 is induced by ν^2, then for every ψ, $\nu^4(\psi) = x$ iff $\nu^2(\mathsf{val}(\psi, x)) = 1$.*

3.2. QBFs and Signed QBFs

We now extend the language $\mathscr{L}$ (respectively, $\mathscr{L}^{\pm}$) with quantifiers $\forall, \exists$ over propositional variables. We denote the extended language by $\mathscr{L}_{\mathsf{Q}}$ (respectively, $\mathscr{L}^{\pm}_{\mathsf{Q}}$). The elements of $\mathscr{L}_{\mathsf{Q}}$ are called quantified Boolean formulas (QBFs), and the elements of $\mathscr{L}^{\pm}_{\mathsf{Q}}$ are called *signed* QBFs. Intuitively, the meaning of a QBF of the form $\exists p \forall q\, \psi$ is that there exists a truth assignment of p such that for every truth assignment of q, ψ is true. Clearly, QBFs can be seen as a conservative extension of propositional formulas. To formalize this, consider a QBF Ψ over $\mathscr{L}_{\mathsf{Q}}$. An occurrence of an atom p in Ψ is called *free* if it is not in the scope of a quantifier $\mathsf{Q}p$, for $\mathsf{Q} \in \{\forall, \exists\}$. We denote by $\Psi[\phi_1/p_1, \ldots, \phi_n/p_n]$ the uniform substitution of each free occurrence of a variable (atom) p_i in Ψ by a formula ϕ_i, for $i = 1, \ldots, n$. Now, the definition of a valuation can be extended to QBFs as follows:

$$
\begin{array}{ll}
\nu(\neg\psi) = \neg\nu(\psi), & \nu(\psi \circ \phi) = \nu(\psi) \circ \nu(\phi), \text{ where } \circ \in \{\wedge, \vee, \supset\}, \\
\nu(\forall p\, \psi) = \nu(\psi[\mathsf{t}/p]) \wedge \nu(\psi[\mathsf{f}/p]), & \nu(\exists p\, \psi) = \nu(\psi[\mathsf{t}/p]) \vee \nu(\psi[\mathsf{f}/p]).
\end{array}
$$

As before, we say that a (two-valued) valuation ν *satisfies* a QBF Ψ if $\nu(\Psi) = 1$ and that ν is a *model* of a set Γ of QBFs if ν satisfies every element of Γ.

4. Logic-Based Argumentation Semantics

We are now ready to use signed QBF-based theories for representing and reasoning with the various semantics of argumentation systems, as depicted in Definitions 3 and 7.

4.1. Complete Semantics

By Definition 6, complete extensions may be represented by a three-valued semantics, where in, out, undec correspond, respectively, to t, f, $\perp$. The next definition reflects this.

Definition 15 Given an AF $\mathscr{A} = \langle Ar, att\rangle$, we denote by $\mathsf{LAB}_{\mathscr{A}}(x)$ the following set of expressions:

$$\left\{ \begin{array}{l} \mathsf{val}(x,t) \supset \bigwedge_{y\in Ar}\Big(\mathsf{att}(y,x) \supset \mathsf{val}(y,f)\Big), \\ \mathsf{val}(x,f) \supset \bigvee_{y\in Ar}\Big(\mathsf{att}(y,x) \wedge \mathsf{val}(y,t)\Big), \\ \mathsf{val}(x,\perp) \supset \Big(\neg\bigwedge_{y\in Ar}\Big(\mathsf{att}(y,x) \supset \mathsf{val}(y,f)\Big) \wedge \neg\bigvee_{y\in Ar}\Big(\mathsf{att}(y,x) \wedge \mathsf{val}(y,t)\Big)\Big) \end{array} \right\}.$$

$\mathsf{LAB}_{\mathscr{A}}(x)$ is an abbreviation of the signed theory that is induced by $\mathscr{A} = \langle Ar, att\rangle$. Here, x should be sequentially substituted by the elements of Ar, $\mathsf{val}(x,v)$ are the signed formulas in Definition 13, $\mathsf{att}(y,x)$ is the propositional constant t if $(y,x) \in att$ and otherwise $\mathsf{att}(y,x)$ is the propositional constant f. By this, the formulas in $\mathsf{LAB}_{\mathscr{A}}$ represent the three requirements from a complete labeling, specified in Definition 6.

Given an AF $\mathscr{A} = \langle Ar, att\rangle$, we denote by $\mathsf{LAB}_{\mathscr{A}}[A_i/x]$ the expressions in Definition 15, evaluated with respect to the argument $A_i \in Ar$. By sequentially evaluating the expressions of Definition 15 with respect to all the arguments in Ar, we get a signed theory whose propositional variables are $Ar^{\pm} = \{A_i^{\oplus} \mid A_i \in Ar\} \cup \{A_i^{\ominus} \mid A_i \in Ar\}$.[5]

Since we are in the three-valued context, we should prevent $\top$-assignments. As $\nu^4(p) = \top$ iff for the induced valuation ν^2, $\nu^2(p^{\oplus}) = 1$ and $\nu^2(p^{\ominus}) = 1$, the coherence condition on Ar for excluding $\top$-assignments is $\mathsf{COH}(Ar) = \big\{\neg(A_i^{\oplus} \wedge A_i^{\ominus}) \mid A_i \in Ar\big\}$.

Definition 16 Given $\mathscr{A} = \langle Ar, att\rangle$, we let $\mathsf{CMP}(\mathscr{A})$ be $\bigcup_{A_i\in Ar} \mathsf{LAB}_{\mathscr{A}}[A_i/x] \,\cup\, \mathsf{COH}(Ar)$.

Example 17 Consider again the argumentation framework $\mathscr{A}_1$ in Example 4. In this case, $\mathsf{LAB}_{\mathscr{A}_1}$ is the following theory:

$$\begin{array}{ll} \mathsf{val}(A,t) \supset \mathsf{val}(B,f), & \mathsf{val}(A,f) \supset \mathsf{val}(B,t), \\ \mathsf{val}(B,t) \supset \mathsf{val}(A,f), & \mathsf{val}(B,f) \supset \mathsf{val}(A,t), \\ \mathsf{val}(C,t) \supset (\mathsf{val}(B,f) \wedge \mathsf{val}(E,f)), & \mathsf{val}(C,f) \supset (\mathsf{val}(B,t) \vee \mathsf{val}(E,t)), \\ \mathsf{val}(D,t) \supset \mathsf{val}(C,f), & \mathsf{val}(D,f) \supset \mathsf{val}(C,t), \\ \mathsf{val}(E,t) \supset \mathsf{val}(D,f), & \mathsf{val}(E,f) \supset \mathsf{val}(D,t), \\ \mathsf{val}(A,\perp) \supset (\neg\mathsf{val}(B,f) \wedge \neg\mathsf{val}(B,t)), & \\ \mathsf{val}(B,\perp) \supset (\neg\mathsf{val}(A,f) \wedge \neg\mathsf{val}(A,t)), & \\ \mathsf{val}(C,\perp) \supset \neg(\mathsf{val}(B,f) \wedge \mathsf{val}(E,f)) \wedge \neg(\mathsf{val}(B,t) \vee \mathsf{val}(E,t)), & \\ \mathsf{val}(D,\perp) \supset (\neg\mathsf{val}(C,f) \wedge \neg\mathsf{val}(C,t)), & \\ \mathsf{val}(E,\perp) \supset (\neg\mathsf{val}(D,f) \wedge \neg\mathsf{val}(D,t)). & \end{array}$$

[5] Here and in what follows we freely exchange an argument $A_i \in Ar$, the propositional variable that represents A_i (with the same notation), and the corresponding signed variables $A_i^{\oplus}$, $A_i^{\ominus}$ in $\mathsf{LAB}_{\mathscr{A}}$.

Thus, $\mathsf{CMP}(\mathscr{A}_1) = \mathsf{LAB}_{\mathscr{A}_1} \cup \mathsf{COH}(Ar)$ is the following theory:

$$
\begin{array}{ll}
(A^{\oplus} \wedge \neg A^{\ominus}) \supset (B^{\ominus} \wedge \neg B^{\oplus}), & (A^{\ominus} \wedge \neg A^{\oplus}) \supset (B^{\oplus} \wedge \neg B^{\ominus}), \\
(B^{\oplus} \wedge \neg B^{\ominus}) \supset (A^{\ominus} \wedge \neg A^{\oplus}), & (B^{\ominus} \wedge \neg B^{\oplus}) \supset (A^{\oplus} \wedge \neg A^{\ominus}), \\
(C^{\oplus} \wedge \neg C^{\ominus}) \supset ((B^{\ominus} \wedge \neg B^{\oplus}) \wedge (E^{\ominus} \wedge \neg E^{\oplus})), & (C^{\ominus} \wedge \neg C^{\oplus}) \supset ((B^{\oplus} \wedge \neg B^{\ominus}) \vee (E^{\oplus} \wedge \neg E^{\ominus})), \\
(D^{\oplus} \wedge \neg D^{\ominus}) \supset (C^{\ominus} \wedge \neg C^{\oplus}), & (D^{\ominus} \wedge \neg D^{\oplus}) \supset (C^{\oplus} \wedge \neg C^{\ominus}), \\
(E^{\oplus} \wedge \neg E^{\ominus}) \supset (D^{\ominus} \wedge \neg D^{\oplus}), & (E^{\ominus} \wedge \neg E^{\oplus}) \supset (D^{\oplus} \wedge \neg D^{\ominus}),
\end{array}
$$

$$
\begin{array}{l}
(\neg A^{\oplus} \wedge \neg A^{\ominus}) \supset (\neg(B^{\ominus} \wedge \neg B^{\oplus}) \wedge \neg(B^{\oplus} \wedge \neg B^{\ominus})), \\
(\neg B^{\oplus} \wedge \neg B^{\ominus}) \supset (\neg(A^{\ominus} \wedge \neg A^{\oplus}) \wedge \neg(A^{\oplus} \wedge \neg A^{\ominus})), \\
(\neg C^{\oplus} \wedge \neg C^{\ominus}) \supset \neg((B^{\ominus} \wedge \neg B^{\oplus}) \wedge (E^{\ominus} \wedge \neg E^{\oplus})) \wedge \neg((B^{\oplus} \wedge \neg B^{\ominus}) \vee (E^{\oplus} \wedge \neg E^{\ominus})), \\
(\neg D^{\oplus} \wedge \neg D^{\ominus}) \supset (\neg(C^{\ominus} \wedge \neg C^{\oplus}) \wedge \neg(C^{\oplus} \wedge \neg C^{\ominus})), \\
(\neg E^{\oplus} \wedge \neg E^{\ominus}) \supset (\neg(D^{\ominus} \wedge \neg D^{\oplus}) \wedge \neg(D^{\oplus} \wedge \neg D^{\ominus}),
\end{array}
$$

$$\neg(A^{\oplus} \wedge A^{\ominus}),\ \neg(B^{\oplus} \wedge B^{\ominus}),\ \neg(C^{\oplus} \wedge C^{\ominus}),\ \neg(D^{\oplus} \wedge D^{\ominus}),\ \neg(E^{\oplus} \wedge E^{\ominus}).$$

The two-valued and three-valued models of this theory are the following:

	$A^{\oplus}$	$A^{\ominus}$	$B^{\oplus}$	$B^{\ominus}$	$C^{\oplus}$	$C^{\ominus}$	$D^{\oplus}$	$D^{\ominus}$	$E^{\oplus}$	$E^{\ominus}$
μ_1	1	0	0	1	0	0	0	0	0	0
μ_2	0	1	1	0	0	1	1	0	0	1
μ_3	0	0	0	0	0	0	0	0	0	0

ν	A	B	C	D	E
ν_1	t	f	$\bot$	$\bot$	$\bot$
ν_2	f	t	f	t	f
ν_3	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

The sets of atoms that are assigned true by these valuations are $\{A\}$, $\{B,D\}$, and $\emptyset$. These are exactly the complete extensions of $\mathscr{A}_1$, as indeed indicated in Corollary 19 below.

Proposition 18 *Let $\mathscr{A} = \langle Ar, att\rangle$ be an AF. Then there is a one-to-one correspondence between the elements of (1) the complete extensions of $\mathscr{A}$, (2) the complete labelings of $\mathscr{A}$, and (3) the models of $\mathsf{CMP}(\mathscr{A})$.*[6]

Corollary 19 *Let $\mathscr{A} = \langle Ar, att\rangle$ be an AF. Then E is a complete extension of $\mathscr{A}$ iff there is a three-valued valuation ν^3 that is associated with a model ν of $\mathsf{CMP}(\mathscr{A})$ such that:*

- $E = \mathsf{In}(\nu) = \{A_i \in Ar \mid \nu^3(A_i) = t\}$,
- $E^+ = \mathsf{Out}(\nu) = \{A_i \in Ar \mid \nu^3(A_i) = f\}$,
- $Ar \setminus (E \cup E^+) = \mathsf{Undec}(\nu) = \{A_i \in Ar \mid \nu^3(A_i) = \bot\}$.

4.2. Stable Semantics

By Definition 3, a stable extension of an argumentation framework $\mathscr{A} = \langle Ar, att\rangle$ is a complete extension E of $\mathscr{A}$ such that $E \cup E^+ = Ar$. It follows, then, that:

Proposition 20 *Let $\mathscr{A}$ be an AF. Then E is a stable extension of $\mathscr{A}$ iff there is a model ν of $\mathsf{CMP}(\mathscr{A})$ such that $\mathsf{In}(\nu) = E$, $\mathsf{Out}(\nu) = E^+$, and $\mathsf{Undec}(\nu) = \emptyset$.*

The last proposition can be represented by a corresponding signed theory as follows:

Definition 21 Given an AF $\mathscr{A} = \langle Ar, att\rangle$, we denote $\mathsf{SE}(\mathscr{A}) = \mathsf{CMP}(\mathscr{A}) \cup \mathsf{EM}(Ar)$, where $\mathsf{EM}(Ar)$ 'excludes the middle-value' ($\bot$), i.e., $\mathsf{EM}(Ar) = \{(A_i^{\oplus} \vee A_i^{\ominus}) \mid A_i \in Ar\}$.

[6]Due to lack of space proofs are omitted.

Proposition 22 *Let $\mathscr{A} = \langle Ar, att\rangle$ be an AF, and let E be a complete extension of $\mathscr{A}$.*

- *E is a stable extension of $\mathscr{A}$ iff there is a two-valued model ν^2 of $\mathsf{SE}(\mathscr{A})$ such that $E = \mathsf{In}(\nu^2)$ and $E^+ = \mathsf{Out}(\nu^2)$,*
- *E is a stable extension of $\mathscr{A}$ iff there is a 3-valued valuation ν^3 that is associated with a model of $\mathsf{SE}(\mathscr{A})$ s.t. $E = \{A_i \mid \nu^3(A_i) = t\}$, and $E^+ = \{A_i \mid \nu^3(A_i) = f\}$.*

Example 23 Consider the theory $\mathsf{CMP}(\mathscr{A}_1)$ of Example 17. Since only μ_2 satisfies $\mathsf{EM}(Ar_1)$, it is the only model of $\mathsf{SE}(\mathscr{A}_1)$. Now, since $\{B,D\} = \mathsf{In}(\mu_2) = \{x \mid \nu_2(x) = t\}$, it follows that $\{B,D\}$ is the only stable extension of $\mathscr{A}_1$, as guaranteed by Proposition 22.

4.3. Semi-Stable Semantics

Recall that a semi-stable extension of $\mathscr{A} = \langle Ar, att\rangle$ is a complete extension E of $\mathscr{A}$ for which the set $E \cup E^+$ is maximal, i.e., $Ar \setminus (E \cup E^+)$ is *minimal*. Thus, for representing the semi-stable extensions of $\mathscr{A}$ we have to identify the models of $\mathsf{CMP}(\mathscr{A})$ and 'filter out' those models that do not minimize the $\perp$-assignments. In other words, we have to compute the $\leq_\perp$-minimal models of $\mathsf{CMP}(\mathscr{A})$, where:

- for 2-valued valuations ν, μ on $Ar^\pm$, $\nu \leq_\perp \mu$ iff $\mathsf{Undec}(\nu) \subseteq \mathsf{Undec}(\mu)$,
- for 3-valued valuations ν, μ on Ar, $\nu \leq_\perp \mu$ iff $\{A_i \mid \nu(A_i) = \perp\} \subseteq \{A_i \mid \mu(A_i) = \perp\}$.

Next, we represent the $\leq_\perp$-minimal models of $\mathsf{CMP}(\mathscr{A})$ by augmenting $\mathsf{CMP}(\mathscr{A})$ with a condition (a circumscriptive-like QBF) that assures minimization of $\perp$-assignments.

Definition 24 Let $\mathscr{A} = \langle Ar, att\rangle$ be an AF with $|Ar| = n$. We denote by $\psi[p_i^\pm/A_i^\pm]$ the formula $\psi[p_1^\oplus/A_1^\oplus, p_1^\ominus/A_1^\ominus, \ldots, p_n^\oplus/A_n^\oplus, p_n^\ominus/A_n^\ominus]$ and abbreviate by $\mathsf{CMP}(\mathscr{A})_\wedge$ the conjunction of the formulas in $\mathsf{CMP}(\mathscr{A})$. Let $\mathsf{Min}_{\leq_\perp}(\mathsf{CMP}(\mathscr{A}))$ be the following QBF:

$$\forall p_1^\oplus, p_1^\ominus, \ldots, p_n^\oplus, p_n^\ominus \Big(\ \mathsf{CMP}(\mathscr{A})_\wedge \Big[p_i^\pm/A_i^\pm\Big] \supset \\ \Big(\bigwedge_{A_i \in Ar} \Big(\mathsf{val}(A_i,\perp)\Big[p_i^\pm/A_i^\pm\Big] \supset \mathsf{val}(A_i,\perp)\Big) \supset \bigwedge_{A_i \in Ar} \Big(\mathsf{val}(A_i,\perp) \supset \mathsf{val}(A_i,\perp)\Big[p_i^\pm/A_i^\pm\Big]\Big)\Big)\Big).$$

Definition 25 Given $\mathscr{A} = \langle Ar, att\rangle$, denote: $\mathsf{SSE}(\mathscr{A}) = \mathsf{CMP}(\mathscr{A}) \cup \{\mathsf{Min}_{\leq_\perp}(\mathsf{CMP}(\mathscr{A}))\}$.

Proposition 26 *Let $\mathscr{A} = \langle Ar, att\rangle$ be an AF, and E a complete extension of $\mathscr{A}$. Then E is a semi-stable extension of $\mathscr{A}$ iff there is a 3-valued valuation ν^3 that is associated with a model of $\mathsf{SSE}(\mathscr{A})$ s.t. $E = \{A_i \in Ar \mid \nu^3(A_i) = t\}$ and $E^+ = \{A_i \in Ar \mid \nu^3(A_i) = f\}$.*

4.4. Grounded and Preferred Semantics

Grounded extensions and preferred extensions of an argumentation framework $\mathscr{A}$ can be represented like the semi-stable extensions of $\mathscr{A}$, but this time, for representing preferred (respectively, grounded) extensions of $\mathscr{A}$, we have to augment $\mathsf{CMP}(\mathscr{A})$ with a criterion that assures maximality (respectively, minimality) with respect to the following orders:

- for 2-valued valuations ν, μ on $Ar^\pm$, $\nu \leq_t \mu$ iff $\mathsf{In}(\nu) \subseteq \mathsf{In}(\mu)$,
- for 3-valued valuations ν, μ on Ar, $\nu \leq_t \mu$ iff $\{A_i \mid \nu(A_i) = t\} \subseteq \{A_i \mid \mu(A_i) = t\}$.

Again, this can be done by corresponding QBFs. Minimization of t-assignments among the valuations that are associated with the models of $\mathsf{CMP}(\mathscr{A})$ can be specified by a QBF, denoted $\mathsf{Min}_{\leq_t}(\mathsf{CMP}(\mathscr{A}))$, that is obtained from $\mathsf{Min}_{\leq_\perp}(\mathsf{CMP}(\mathscr{A}))$ (Definition 24) by replacing every occurrence of $\mathsf{val}(A_i,\perp)$ with the signed formula $\mathsf{val}(A_i,t)$. Similarly, maximization of t-assignments can be specified by the following QBF, denoted $\mathsf{Max}_{\leq_t}(\mathsf{CMP}(\mathscr{A}))$:

$$\forall p_1^{\oplus}, p_1^{\ominus}, \ldots, p_n^{\oplus}, p_n^{\ominus} \Bigg(\mathsf{CMP}(\mathscr{A})_\wedge \Big[p_i^{\pm}/A_i^{\pm}\Big] \supset$$
$$\Big(\bigwedge_{A_i \in Ar} \Big(\mathsf{val}(A_i,t) \supset \mathsf{val}(A_i,t)\Big[p_i^{\pm}/A_i^{\pm}\Big]\Big) \supset \bigwedge_{A_i \in Ar} \Big(\mathsf{val}(A_i,t)\Big[p_i^{\pm}/A_i^{\pm}\Big] \supset \mathsf{val}(A_i,t)\Big)\Big) \Bigg).$$

Definition 27 Given $\mathscr{A} = \langle Ar, att\rangle$, denote: $\mathsf{GE}(\mathscr{A}) = \mathsf{CMP}(\mathscr{A}) \cup \{\mathsf{Min}_{\leq_t}(\mathsf{CMP}(\mathscr{A}))\}$ and $\mathsf{PE}(\mathscr{A}) = \mathsf{CMP}(\mathscr{A}) \cup \{\mathsf{Max}_{\leq_t}(\mathsf{CMP}(\mathscr{A}))\}$.

Proposition 28 *Let $\mathscr{A} = \langle Ar, att\rangle$ be an AF, and let E be a complete extension of $\mathscr{A}$.*

- *E is the grounded extension of $\mathscr{A}$ iff there is a three-valued valuation ν^3 that is associated with a model of $\mathsf{GE}(\mathscr{A})$ such that $E = \{A_i \in Ar \mid \nu^3(A_i) = t\}$ and $E^+ = \{A_i \in Ar \mid \nu^3(A_i) = f\}$.*
- *E is a preferred extension of $\mathscr{A}$ iff there is a three-valued valuation ν^3 that is associated with a model of $\mathsf{PE}(\mathscr{A})$ such that $E = \{A_i \in Ar \mid \nu^3(A_i) = t\}$ and $E^+ = \{A_i \in Ar \mid \nu^3(A_i) = f\}$.*

Example 29 Consider the signed theory $\mathsf{CMP}(\mathscr{A}_1)$ of Example 17. Among the three models of $\mathsf{CMP}(\mathscr{A}_1)$, μ_3 satisfies $\mathsf{Min}_{\leq_t}(\mathsf{CMP}(\mathscr{A}_1))$ and both of μ_1 and μ_2 satisfy $\mathsf{Max}_{\leq_t}(\mathsf{CMP}(\mathscr{A}_1))$. Thus, $mod(\mathsf{GE}(\mathscr{A}_1)) = \{\mu_3\}$ and $mod(\mathsf{PE}(\mathscr{A}_1)) = \{\mu_1, \mu_2\}$. In the notations of Example 17, then,

a) ν_3 is the only three-valued valuation that is relevant for the first item of Proposition 28, and so $\{x \mid \nu_3(x) = t\} = \emptyset$ is the grounded extension of $\mathscr{A}_1$.

b) ν_1 and ν_2 are the three-valued valuations that are relevant for the second item of Proposition 28, and so both $\{x \mid \nu_1(x) = t\} = \{A\}$ and $\{x \mid \nu_2(x) = t\} = \{B, D\}$ are the preferred extensions of $\mathscr{A}_1$.

4.5. Ideal and Eager Semantics

Using the signed QBF theory PE for representing preferred extensions, it is possible to represent the ideal extension as well.

Definition 30 Let $\mathscr{A} = \langle Ar, att\rangle$ with $|Ar| = n$. Denote by $\mathsf{SubSet}_{\leq_t}(\mathsf{PE}(\mathscr{A}))$ the QBF: $\forall q_1^{\oplus}, q_1^{\ominus}, \ldots, q_n^{\oplus}, q_n^{\ominus} \Big(\mathsf{PE}(\mathscr{A})_\wedge \Big[p_i^{\pm}/A_i^{\pm}\Big] \supset \bigwedge_{A_i \in Ar} \Big(\mathsf{val}(A_i,t) \supset \mathsf{val}(A_i,t)\Big[p_i^{\pm}/A_i^{\pm}\Big]\Big) \Big)$. Define:

$$\mathsf{PreIE}(\mathscr{A}) = \mathsf{CMP}(\mathscr{A}) \cup \{\mathsf{SubSet}_{\leq_t}(\mathsf{PE}(\mathscr{A}))\},$$
$$\mathsf{IE}(\mathscr{A}) = \mathsf{PreIE}(\mathscr{A}) \cup \{\mathsf{Max}_{\leq_t}(\mathsf{PreIE}(\mathscr{A}))\},$$

where $\mathsf{Max}_{\leq_t}(\mathsf{PreIE}(\mathscr{A}))$ is obtained from $\mathsf{Max}_{\leq_t}(\mathsf{CMP}(\mathscr{A}))$ by substituting $\mathsf{CMP}(\mathscr{A})_\wedge$ by $\mathsf{PreIE}(\mathscr{A})_\wedge$.

In terms of labeling functions, PreIE (denoting 'pre-ideal' extensions) states that the labeling has to be a complete one, and its set of in-labeled arguments should be a subset of each set of in-labeled arguments of the preferred labelings of $\mathscr{A}$. In turn, IE selects among these (pre-ideal) labelings the one with maximal in-labeled arguments (i.e., the in-maximal pre-ideal set). Hence, $\mathsf{IE}(\mathscr{A})$ selects the ideal labeling of $\mathscr{A}$. Thus,

Proposition 31 *Let $\mathscr{A}$ be an argumentation framework. Then E is the ideal extension of $\mathscr{A}$ iff there is a model ν of $\mathsf{IE}(\mathscr{A})$ such that $\mathsf{In}(\nu) = E$ and $\mathsf{Out}(\nu) = E^+$.*

Eager semantics is defined like ideal semantics, but with respect to semi-stable extensions instead of preferred extensions. So in order to represent the eager extension we just have to replace in Definition 30 the signed QBF theory PE, representing preferred extensions, by the signed QBF theory SSE, representing semi-stable extensions. Thus,

Definition 32 Given an AF $\mathscr{A} = \langle Ar, att \rangle$, we denote

$$\mathsf{PreEE}(\mathscr{A}) \;=\; \mathsf{CMP}(\mathscr{A}) \,\cup\, \{\mathsf{SubSet}_{\leq_t}(\mathsf{SSE}(\mathscr{A}))\},$$
$$\mathsf{EE}(\mathscr{A}) \;=\; \mathsf{PreEE}(\mathscr{A}) \,\cup\, \{\mathsf{Max}_{\leq_t}(\mathsf{PreEE}(\mathscr{A}))\}.$$

$\mathsf{SubSet}_{\leq_t}(\mathsf{SSE}(\mathscr{A}))$ is obtained from $\mathsf{SubSet}_{\leq_t}(\mathsf{PE}(\mathscr{A}))$ by substituting $\mathsf{PE}(\mathscr{A})_\wedge$ by $\mathsf{SSE}(\mathscr{A})_\wedge$. $\mathsf{Max}_{\leq_t}(\mathsf{PreEE}(\mathscr{A}))$ is obtained from $\mathsf{Max}_{\leq_t}(\mathsf{CMP}(\mathscr{A}))$ by substituting $\mathsf{CMP}(\mathscr{A})_\wedge$ by $\mathsf{PreEE}(\mathscr{A})_\wedge$.

Similar considerations as before imply that PreEE represents the 'pre-eager' extensions of $\mathscr{A}$ (i.e., the complete labelings of $\mathscr{A}$ whose set of in-labeled arguments is a subset of the set of in-labeled arguments of every semi-stable labeling of $\mathscr{A}$), and EE represents the 'pre-eager' labeling with maximal in-assignments. Thus, $\mathsf{EE}(\mathscr{A})$ represents the eager extension of $\mathscr{A}$.

Proposition 33 *Let $\mathscr{A}$ be an argumentation framework. Then E is the eager extension of $\mathscr{A}$ iff there is a model ν of $\mathsf{EE}(\mathscr{A})$ such that $\mathsf{In}(\nu) = E$ and $\mathsf{Out}(\nu) = E^+$.*

4.6. Stage Semantics

The definition of stage extensions resembles that of semi-stable extensions. Both extensions are sets of arguments with maximal range, but in contrast to semi-stable extensions, for representing stage extensions we need a signed theory that formalizes conflict-free labelings. This is what we do next.

Definition 34 Given $\mathscr{A} = \langle Ar, att \rangle$, we denote by $\mathsf{CFLAB}_{\mathscr{A}}(x)$ the following set:

$$\left\{ \begin{array}{l} \mathsf{val}(x,t) \supset \bigwedge_{y \in Ar} \big(\mathsf{att}(y,x) \supset \neg \mathsf{val}(y,t)\big), \\ \mathsf{val}(x,f) \supset \bigvee_{y \in Ar} \big(\mathsf{att}(y,x) \wedge \mathsf{val}(y,t)\big) \end{array} \right\}.$$

The meanings of the expressions above are similar to those in Definition 15. By this, $\mathsf{CFLAB}_{\mathscr{A}}$ represents the two requirements **(Pos1)** and **(Neg)** of a conflict-free labeling, given in Definition 6. Again, we denote by $\mathsf{CFLAB}_{\mathscr{A}}[A_i/x]$ the expressions of Definition 34, evaluated w.r.t. $A_i \in Ar$.

Definition 35 Given $\mathscr{A} = \langle Ar, att \rangle$, we let $\mathsf{CF}(\mathscr{A})$ be $\bigcup_{A_i \in Ar} \mathsf{CFLAB}_{\mathscr{A}}[A_i/x] \cup \mathsf{COH}(Ar)$.

Proposition 36 *There is a one-to-one correspondence between the conflict-free labelings of an argumentation framework* $\mathscr{A}$ *and the models of* $\mathsf{CF}(\mathscr{A})$.

Definition 37 Given $\mathscr{A} = \langle Ar, att \rangle$, we let $\mathsf{SGE}(\mathscr{A})$ be $\mathsf{CF}(\mathscr{A}) \cup \{\mathsf{Min}_{\leq_\perp}(\mathsf{CF}(\mathscr{A}))\}$.

Proposition 38 *Let* $\mathscr{A} = \langle Ar, att \rangle$ *be an AF. A subset E of Ar is a stage extension of* $\mathscr{A}$ *iff there is a three-valued valuation* ν^3 *that is associated with a model* ν^2 *of* $\mathsf{SGE}(\mathscr{A})$, *such that* $E = \mathsf{In}(\nu^2) = \{A_i \in Ar \mid \nu^3(A_i) = t\}$ *and* $E^+ = \mathsf{Out}(\nu^2) = \{A_i \in Ar \mid \nu^3(A_i) = f\}$.

5. Summary and Perspective

Table 1 summarizes the one-to-one correspondence between extension-based semantics, argumentation labelings, and models of signed (QBF) theories, as depicted in this paper.

extension	**labeling**	**signed (QBF) theory**		
complete	complete	CMP		Def. 16
stable	complete without undec	SE	$[\mathsf{CMP} + \mathsf{EM}]$	Def. 21
semi-stable	complete with minimal undec	SSE	$[\mathsf{CMP} + \mathsf{Min}_{\leq_\perp}(\mathsf{CMP})]$	Def. 25
preferred	complete with maximal in	PE	$[\mathsf{CMP} + \mathsf{Max}_{\leq_t}(\mathsf{CMP})]$	Def. 27
grounded	complete with minimal in	GE	$[\mathsf{CMP} + \mathsf{Min}_{\leq_t}(\mathsf{CMP})]$	Def. 27
pre-ideal	complete with in-subset w.r.t. preferred	PreIE	$[\mathsf{CMP} + \mathsf{SubSet}_{\leq_t}(\mathsf{PE})]$	Def. 30
ideal	pre-ideal with maximal in	IE	$[\mathsf{PreIE} + \mathsf{Max}_{\leq_t}(\mathsf{PreIE})]$	Def. 30
pre-eager	complete with in-subset w.r.t. semi-stable	PreEE	$[\mathsf{CMP} + \mathsf{SubSet}_{\leq_t}(\mathsf{SSE})]$	Def. 32
eager	pre-eager with maximal in	EE	$[\mathsf{PreEE} + \mathsf{Max}_{\leq_t}(\mathsf{PreEE})]$	Def. 32
stage	conflict-free with minimal undec	SGE	$[\mathsf{CF} + \mathsf{Min}_{\leq_\perp}(\mathsf{CF})]$	Def. 37

Table 1. The relations among the three approaches to abstract argumentation semantics

A logic-based analysis of argumentation semantics is also provided by [6], where notions like a preferred, complete or grounded extension are taken as primitives. Other frameworks based on modal logics are described in [12] and in [17,18]. In comparison to these works, we note that the incorporation of QBFs and multiple-valued logics allows us to uniformly represent a wide range of semantics, using simpler languages and stricter approach. An early approach of applying QBFs to model argumentation problems appears in [16], using assumption-based argumentation [7] instead of Dung's abstract argumentation frameworks.

Some of the most expressive approaches, not only for characterizing argumentation semantics but also for computing them, have been stated in the field of logic programming. (see, e.g., [19] for an overview). The currently most advanced approach in this area is the answer set programming application `ASPARTIX` [15] that is able to compute a wide range of argumentation semantics without the need to apply meta-logic programs.

An obvious benefit of the approaches based on pure logic, including the present one, is that they allow to borrow standard and well-studied notions, notations, techniques and results from formal logic, and apply them in the context of argumentation theory. This includes, among others, skeptical acceptance that is reducible to logical entailment,[7] credulous acceptance that is reducible to satisfiability checking,[8] corresponding complexity results, and automated verification of semantical properties, like existence of extensions.

To conclude, our framework provides a uniform and concise way of representing some of the most common extension-based semantics of abstract argumentation theory. This representation also yields an easy way of encoding the semantics of argumentation frameworks by off-the-shelf QBF solvers (see, e.g., `http://www.qbflib.org/`). Whether these methods provide workable solutions for realistic problems can only be determined by implementation and testing. This is a subject for future work.

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[7] E.g., A is skeptically accepted with respect to the semi-stable semantics of $\mathscr{A}$ iff $\mathsf{SSE}(\mathscr{A}) \vdash \mathsf{val}(A,t)$.

[8] E.g., A is credulously accepted with respect to the semi-stable semantics of $\mathscr{A}$ iff $\mathsf{SSE}(\mathscr{A}) \cup \{\mathsf{val}(A,t)\} \not\vdash \mathsf{f}$.